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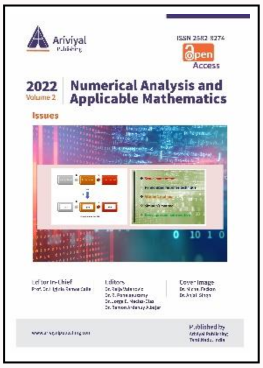

Three Parameter Weighted Exponential-Lomax Distribution: Properties and Applications in Reliability Engineering

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Abstract: In this article, a new distribution named weighted exponential Lomax (WEL) distribution is developed. Probability density function and cumulative distribution function are constructed along with their graphs. Several statistical properties: moments, quantile function, random number generator, mean deviation and reliability measures (survival function, hazard function, cumulative hazard rate, reversed hazard rate and mean residual function), Lorenz & Bonferroni curves, Renyi entropy, Q-entropy and Shannon entropy have also been derived. Parameters are estimated by maximum likelihood estimation (MLE). The model is applied on the two data sets to check the reliability of the products. Finally, after applications and comparison with existing models, it is suggested that the proposed is more flexible in the reliability engineering.

Keywords: WEL; Moments; MGF; CGF; CF; Mean deviation; Entropies; MLE

1. Introduction

Pareto type II is also referred as Lomax distribution was introduced in 1954 by Lomax.[1] Lomax distribution is applied in modeling the data in various fields such as in business used to model data for the income and wealth inequality, in actuarial science to model the data of finance and insurance, in biological science applied for the bladder cancer data and in reliability engineering is used for life testing purposes. Harris et al.[2] used Lomax distribution for queue service discipline. Atkinson, et al.[3] applied to model the wealth and income data. They compare the income ratio before the World War I and after Second World War. Hassan, et al.[4] illustrated the numerical study to discussed optimum step stress Accelerated life testing for Lomax distribution. Bindu and Sangita[5] applied the double Lomax distribution to model the distribution of a cDNA dual dye microarray gene. Al-Zahrani and Al-Harbi[6] estimated the Parameters of Lomax distribution under General Progressive Censoring. Ahsanullah, et. al.[7] established record values from the classical Pareto distribution. Durgamamba et al.[8] determined the reliability test plans for the testing the failure lifetime of products follow a weighted Lomax distribution. Tahir et al.[9] introduced Weibull-Lomax distribution and applied the model on Failure times of Aircraft Windshield. Gupta et al.[10] proposed the Lomax-Gumbel distribution with statistical properties and estimation. Kilany[11] weighted the Lomax distribution and used the Bladder Cancer patient's data for the application purposes. Some more distributions and their structural properties are derived Lomax-Exponentiated distribution by Golzar et at.,[12] Five parameters Beta Lomax distribution by Rajab et al.,[13] Exponentiated- Lomax distribution by Salem,[14] five parameter Lomax distribution by Mead,[15] and The Half-Logistic Lomax distribution for Life modelling by Anwar and Masood.[16]

In this article, three parameter weighted exponential Lomax (WEL) distribution is introduced which is a blend of weighted exponential and Lomax distribution. The same technique of combining various distributions with Lomax distribution was used by Tahir et al.,[18] El-Bassiouny et al.,[19] Shabbir et al.,[20] and Fatima et al.[21]

The pdf of the weighted exponential distribution:

$$f_w(x) = \lambda^2 x e^{-\lambda x}, x > 0, \lambda > 0 \tag{1}$$

where, λ is a scale parameter.

In 1954, Lomax distribution was introduced. Its model belongs to the family of decreasing failure rate. Its probability density function has two parameters and heavy-tailed shape.

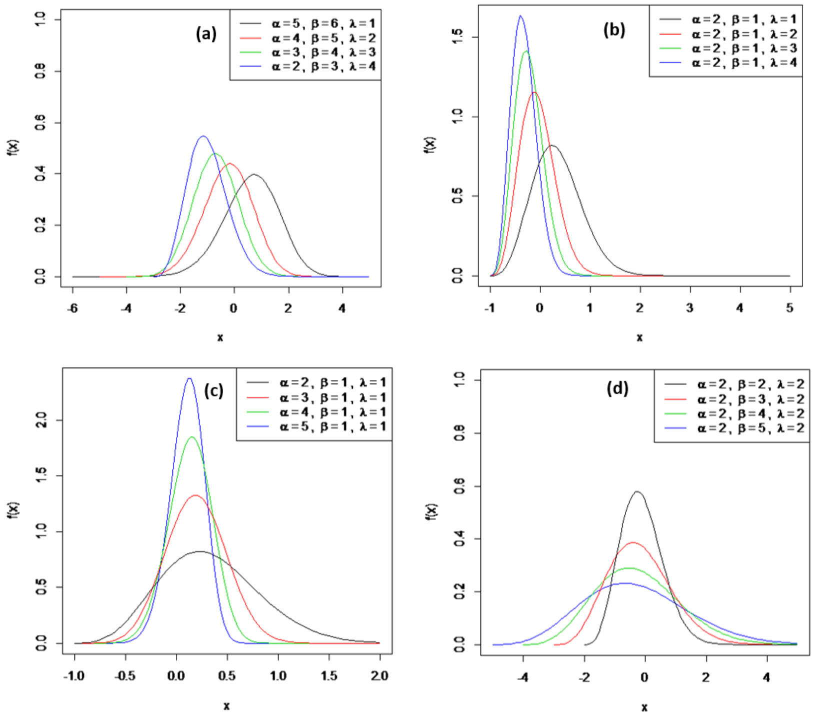

Fig. 1. pdf graphs of WEL distribution for various values of parameters

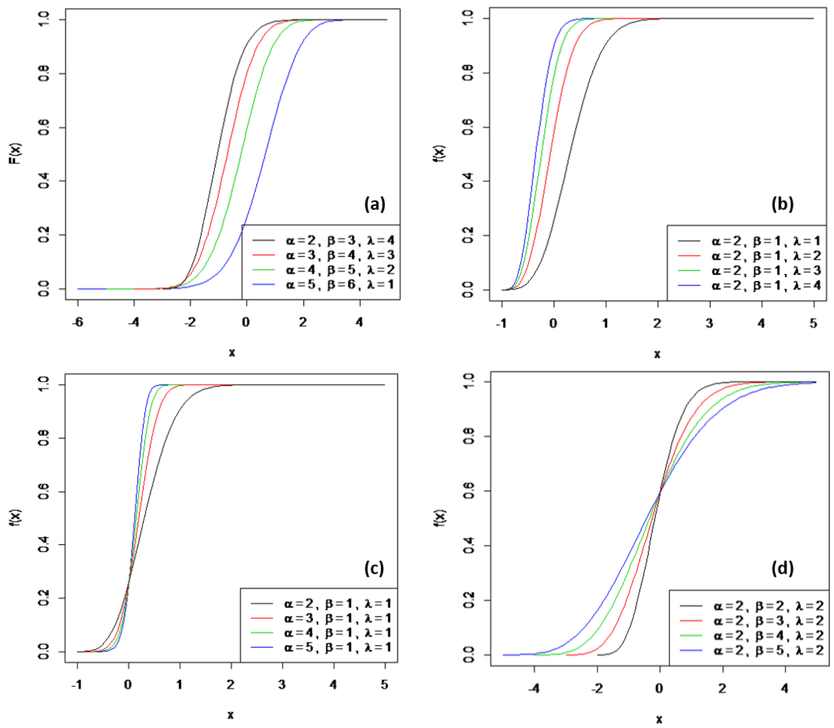

Fig. 2. cdf graphs of WEL distribution for various values for parameters

The pdf and cdf of Lomax distribution is given by:

$$f(x) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \tag{2}$$

$$F(x) = 1-\left[\frac{\beta}{x+\beta}\right]^{\alpha}, x>0, \beta>0 \tag{3}$$

where α is the shape parameter and β is the scale parameter.

Al-Kadim and Boshi[22] gave the criteria of formulation of a model. Following is the method to construct a new distribution:

$$F(x) = \int_0^{\frac{1}{1-F(x)}} f(x)dx \tag{4}$$

2. Weighted Exponential-Lomax (WEL) Distribution

In this paper the new distribution is developed by substituting equation (1) & (3) in (4), we get the cdf of the WEL distribution (Fig. 2) as follows:

$$F(x) = 1 - e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}} \left[1 + \lambda \left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right] \tag{5}$$

the pdf of WEL distribution:

$$f(x) = \frac{\lambda^2\alpha}{\beta}\left[\frac{\beta}{x+\beta}\right]^{-2\alpha+1} e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}} \quad \text{for } x \geq -\beta \tag{6}$$

where λ is the location parameter, α is the scale parameter and β is the shape parameter.

Lemma 2.1: The limits cdf of WEL distribution for $x \to +\infty$ is 1, for $x \to -\beta$ is 0.
Proof: Using equation (5)

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left[1 - e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}\left[1 + \lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right]\right] = 1$$

$$\lim_{x\to-\beta} F(x) = \lim_{x\to-\beta}\left[1 - e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}\left[1 + \lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right]\right] = 0$$

Lemma 2.2: The limits pdf of WELD for $x \to +\infty$ is 0, for $x \to -\beta$ is 0.
Proof: Using equation (6)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{\lambda^2\alpha}{\beta}\left[\frac{\beta}{x+\beta}\right]^{-2\alpha+1} e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}\right] = 0$$

$$\lim_{x\to-\beta} f(x) = \lim_{x\to-\beta}\left[\frac{\lambda^2\alpha}{\beta}\left[\frac{\beta}{x+\beta}\right]^{-2\alpha+1} e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}\right] = 0$$

From Fig. 1, it can be observed that parameters $\lambda, \alpha, \& \beta$ of the WEL distribution are location, scale and shape parameters respectively. In (b) changing λ and fixing α, β the location of the distribution is changed. In (c) changing α and fixing λ, β the scale of the distribution is changed. In (d) changing β and fixing α, λ the shape of the distribution is changed.

3. Statistical Properties

In this section, various statistical properties of WEL distribution are studied:

$$E(x-\mu)^r = \sum_{k=0}^{r}\binom{r}{k}\left(\frac{\beta}{\lambda^{1\backslash\alpha}}\right)^r(-\beta-\mu)^{r-k}\sqrt{\frac{1}{\alpha}+2}, \qquad k = 1,2,3\ldots \tag{7}$$

The r^th^ moment about mean for WEL distribution is given by:

$$E(x)^r = \sum_{k=0}^{r}\binom{r}{k}\left(\frac{\beta}{\lambda^{1\backslash\alpha}}\right)^r(-\beta)^{r-k}\sqrt{\frac{1}{\alpha}+2} \tag{8}$$

Moment generating function (MGF) of the WEL distribution is

$$M_x(t) = \sum_{v=0}^{\infty}\sum_{k=0}^{v}\binom{v}{k}\frac{t^v\beta^v}{\lambda^{k/\alpha}v!}(-1)^{v-k}\sqrt{\frac{k}{\alpha}+2} \tag{9}$$

$$K_x(t) = \log\left(\sum_{v=0}^{\infty}\sum_{k=0}^{v}\binom{v}{k}\frac{t^v\beta^v}{\lambda^{k/\alpha}v!}(-1)^{v-k}\sqrt{\frac{k}{\alpha}+2}\right) \tag{10}$$

The characteristics function of the WEL distribution is

$$\varphi_x(t) = \sum_{v=0}^{\infty}\sum_{k=0}^{v}\binom{v}{k}\frac{(it)^v\beta^v}{\lambda^{k/\alpha}v!}(-1)^{v-k}\sqrt{\frac{k}{\alpha}+2} \tag{11}$$

Equations (7), (8), (9), (10) and (11) can be used to derive moments of the WEL distribution. Therefore, mean, variance, 3^rd^ and 4^th^ moments of the WEL distribution respectively

$$Mean = \mu_1' = \beta\left[\frac{1}{\lambda^{1/\alpha}}\sqrt{\left[\frac{1}{\alpha}+2\right]} - 1\right] \tag{12}$$

$$Variance = \mu_2 = \frac{\beta^2}{\lambda^{2/\alpha}}\left[\sqrt{\left[\frac{2}{\alpha}+2\right]} - \sqrt{\left[\frac{1}{\alpha}+2\right]^2}\right] \tag{13}$$

$$\mu_3 = \frac{\beta^3}{\lambda^{3/\alpha}}\left[\sqrt{\left[\frac{3}{\alpha}+2\right]} - 3\sqrt{\left[\frac{2}{\alpha}+2\right]}\sqrt{\left[\frac{1}{\alpha}+2\right]} + 2\sqrt{\left[\frac{1}{\alpha}+2\right]^3}\right] \tag{14}$$

$$\mu_4 = \frac{\beta^4}{\lambda^{4/\alpha}}\left[\sqrt{\left[\frac{4}{\alpha}+2\right]} - 4\sqrt{\left[\frac{3}{\alpha}+2\right]\sqrt{\left[\frac{1}{\alpha}+2\right]}} + 6\sqrt{\left[\frac{2}{\alpha}+2\right]}\sqrt{\left[\frac{1}{\alpha}+2\right]^2} - 3\sqrt{\left[\frac{1}{\alpha}+2\right]^4}\right] \tag{15}$$

Using equations (12), (13), (14) and (15) we get co-efficient of variation (ρ), Skewness and Kurtosis respectively:

$$\rho = \frac{\sqrt{\frac{1}{\lambda^{2/\alpha}}\left[\sqrt{\left[\frac{2}{\alpha}+2\right]}-\sqrt{\left[\frac{1}{\alpha}+2\right]}^2\right]}}{\left[\frac{1}{\lambda^{1/\alpha}}\sqrt{\left[\frac{1}{\alpha}+2\right]}-1\right]} \tag{16}$$

$$\beta_1 = \frac{\left[\sqrt{\left[\frac{3}{\alpha}+2\right]}-3\sqrt{\left[\frac{2}{\alpha}+2\right]}\sqrt{\left[\frac{1}{\alpha}+2\right]}+2\sqrt{\left[\frac{1}{\alpha}+2\right]}^3\right]}{\left(\left[\sqrt{\left[\frac{2}{\alpha}+2\right]}-\sqrt{\left[\frac{1}{\alpha}+2\right]}\right]^2\right)^{3/2}} \tag{17}$$

$$\beta_2 = \frac{\frac{\beta^4}{\lambda^{4/\alpha}}\left[\sqrt{\left[\frac{4}{\alpha}+2\right]}-4\sqrt{\left[\frac{3}{\alpha}+2\right]}\sqrt{\left[\frac{1}{\alpha}+2\right]}+6\sqrt{\left[\frac{2}{\alpha}+2\right]}\sqrt{\left[\frac{1}{\alpha}+2\right]}^2-3\sqrt{\left[\frac{1}{\alpha}+2\right]}^4\right]}{\left(\frac{\beta^2}{\lambda^{2/\alpha}}\right)\left(\left[\sqrt{\left[\frac{2}{\alpha}+2\right]}-\sqrt{\left[\frac{1}{\alpha}+2\right]}\right]^2\right)^2} \tag{18}$$

Median of WEL distribution is given by

$$m = \beta\left[\frac{1}{\lambda^{1/\alpha}}\left[ln\left[1+\lambda\left[\frac{\beta}{m+\beta}\right]^{-\alpha}\right]-ln\left(\frac{1}{2}\right)\right]^{1/\alpha}-1\right] \tag{19}$$

From equation (19) numerical values for median can be simulated.
Mode of the WEL distribution is

$$Mode = \beta\left(\left[\frac{2\alpha-1}{\lambda\alpha}\right]^{1/\alpha}-1\right) \tag{20}$$

The geometric mean is

$$G.M = \prod_{i=1}^{n}\sum_{k=0}^{\infty}\binom{1/n}{k}(-1)^{\frac{1}{n}-k}\beta^{1/n}\left(\frac{1}{\lambda}\right)^{k/\alpha}\sqrt{\left[\frac{k}{\alpha}+2\right]} \tag{21}$$

The harmonic mean is derived as:

$$H.M = \left[-\frac{1}{\beta}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}\right)^{k/\alpha}\sqrt{\frac{k}{\alpha}+2}\right]^{-1} \tag{22}$$

Mean deviation is obtained by using the following method:

$$D(\mu) = E(|x-\mu|) = \int_0^\infty |x-\mu|\, f(x)dx = \int_0^\mu(\mu-x)f(x)dx + \int_\mu^\infty(x-\mu)f(x)dx = 2\int_0^\mu(\mu-x)f(x)dx$$

$$D(\mu) = 2\mu F(\mu) - 2\int_0^\mu xf(x)dx$$

So the mean deviation for the WEL distribution is;

$$D(\mu) = 2\mu\left[1-e^{-\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}}\left[1+\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}\right]\right]-2\left[\beta\left[\frac{1}{\lambda^{1/\alpha}}\sqrt{\frac{1}{\alpha}+2\gamma_{\frac{1}{\alpha}+2}}\left(\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}\right)-\gamma_2\left(\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}\right)\right]\right] \tag{23}$$

4. Reliability Analysis

In this section, various measures of reliability are derived.
The reliability function of WEL distribution is:

$$R(x) = e^{-\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}}\left[1+\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}\right] \tag{24}$$

The hazard rate of the WEL distribution is:

$$h(x) = \frac{\frac{\lambda^2\alpha}{\beta}\left[\frac{\beta}{x+\beta}\right]^{1-2\alpha}}{\left[1+\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right]} \tag{25}$$

The Reversed Hazard rate of the WEL distribution is

$$r(x) = \frac{\frac{\lambda^2\alpha}{\beta}\left[\frac{\beta}{x+\beta}\right]^{1-2\alpha}}{e^{\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}-\left[1+\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right]} \tag{26}$$

The cumulative hazard rate of the WEL distribution is

$$H(x) = -ln\left[e^{-\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}}\left[1+\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right]\right] \tag{27}$$

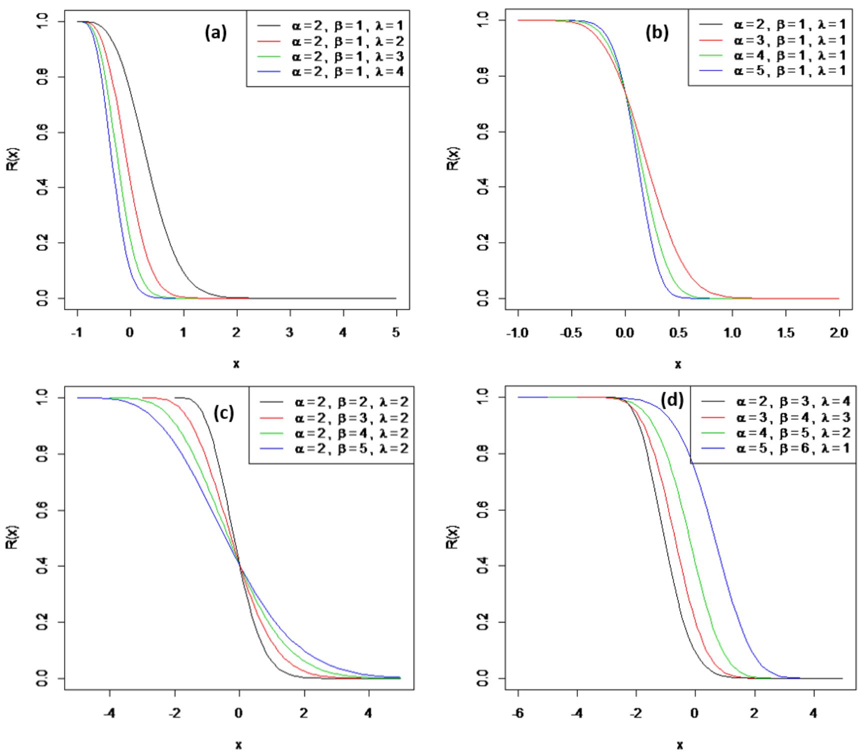

Fig. 3. Reliability function graphs of WEL distribution for various values of parameters

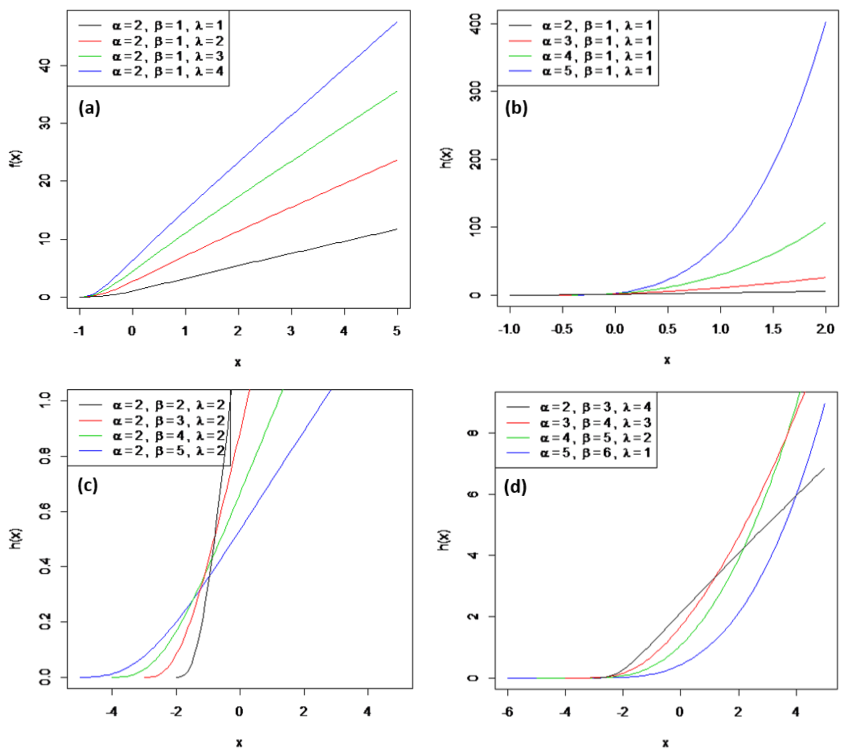

Fig. 4. Hazard rate graphs of WEL distribution for various values of parameters

From fig. 3, it can be seen that the shape for the reliability function graphs of the WEL distribution for values of the parameters is monotonically decreasing. From Fig. 4, it can be seen that the hazard rate of the WEL distribution for various combinations of the parameter values is instantaneously increasing of wear-out failure. Therefore, we can say that WEL distribution having increasing failure rate (IFR). From Fig. 5, it is observed that the in (a), (b) and (d) the reversed hazard rate of the WEL distribution having decreasing trend with infant mortality curve while in (c) it is decreasing monotonically. From fig. 6, it can be seen that the cumulative hazard function of the WEL distribution having increasing trend or increasing monotonically for various combinations of parametric values.

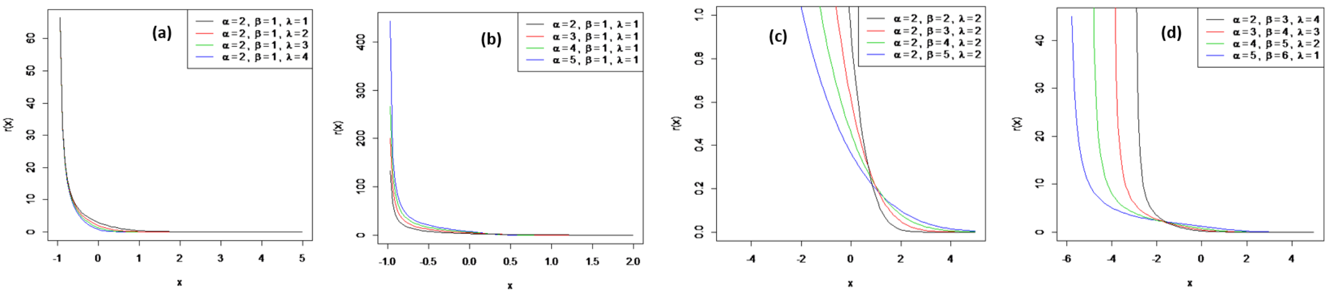

Fig. 5. Reversed hazard graphs of WEL distribution for various values of parameters

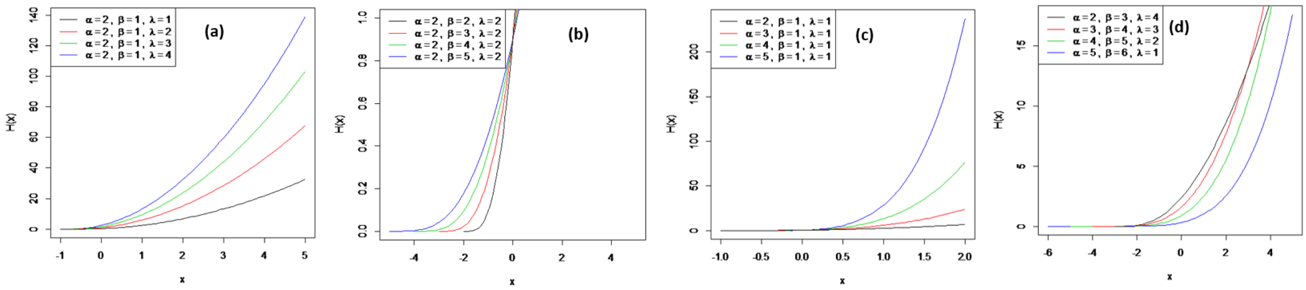

Fig. 6. Cumulative hazard function graphs of WEL distribution for various values of parameters

Theorem 4.1:

Under the following conditions: $H(-\beta) = 0$; $\lim_{x\to+\infty} H(x) = \infty$, the mean residual function of the WEL distribution is

$$m(x) = \frac{\frac{\beta^{\alpha}}{\lambda^{\alpha}}\left(\frac{1}{x+\beta}\right)^{\alpha-1} e^{-\lambda\left[\frac{\beta}{\mu+\beta}\right]^{-\alpha}} + \frac{1}{\lambda^{1/\alpha}}\sqrt{\frac{1}{\alpha}+1}\left(\sqrt{\frac{1}{\alpha}+1}\left(\lambda\left[\frac{\beta}{t+\beta}\right]^{-\alpha}\right)\right)}{e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}\left[1+\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right]} \tag{28}$$

Proof: considering equation (27), we get

$$H(-\beta) = ln\left[e^{-\lambda\left[\frac{-\beta+\beta}{\beta}\right]^{\alpha}}\left[1+\lambda\left[\frac{-\beta+\beta}{\beta}\right]^{\alpha}\right]\right] = 0$$

$$\lim_{x\to\infty} H(x) = \lim_{x\to\infty}\left[-ln\left[e^{-\lambda\left[\frac{-x+\beta}{\beta}\right]^{\alpha}}\left[1+\lambda\left[\frac{\beta}{x+\beta}\right]^{\alpha}\right]\right]\right] = \infty$$

So, the Mean residual function for WEL distribution is

$$m(x) = \int_x^{\infty}\frac{R(t)dt}{R(x)}$$

$$\int_x^{\infty} R(t)dt = \frac{\beta^{\alpha}}{\lambda^{\alpha}}\left(\frac{1}{x+\beta}\right)^{\alpha-1} e^{-\lambda\left[\frac{-\beta+\beta}{\beta}\right]^{\alpha}}\frac{1}{\lambda^{1/\alpha}}\sqrt{\frac{1}{\alpha}+1}\left(\sqrt{\frac{1}{\alpha}+1}\left(\lambda\left[\frac{\beta}{t+\beta}\right]^{-\alpha}\right)\right) \tag{29}$$

By simplifying the equation (29), we get the expression of the mean residual function in equation (27).

5. Quantile Function

Quantile function of WEL distribution is

$$x_q = \beta\left[\frac{1}{\lambda^{1/\alpha}}\left[ln\left[1+\lambda\left[\frac{\beta}{x_q+\beta}\right]^{-\alpha}\right] - ln(1-Q)\right]^{1/\alpha} - 1\right] \tag{30}$$

For q = 0.25, 0.5 and 0.75, we can get the first, second and third quartiles.

5.1. Random Number Generator

Random number generator of WEL distribution is

$$x = \beta\left[\frac{1}{\lambda^{1/\alpha}}\left[ln\left[1+\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right] - ln(1-u)\right]^{1/\alpha} - 1\right] \tag{31}$$

where $u \sim U(0,1)$.

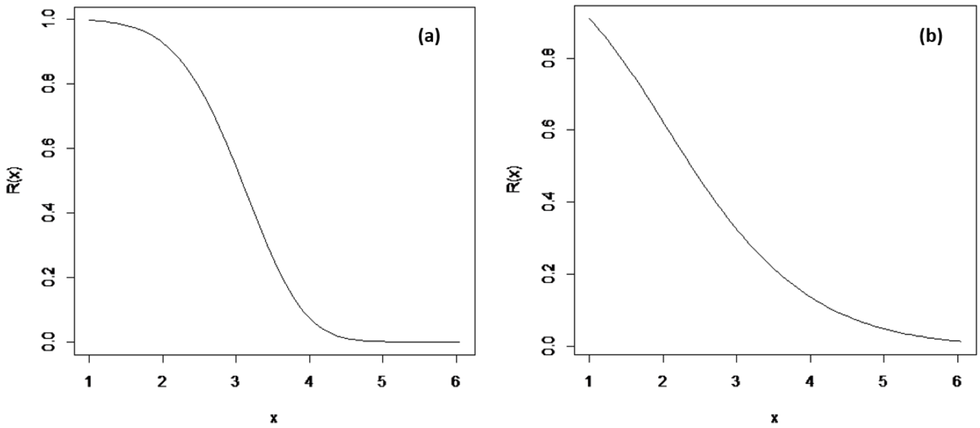

Fig. 7. Reliability function graph for the Gauge length data in (a) and Failure Time of 84 Aircraft Windshield data in (b)

Table 1. MLEs and the Statistics (AIC, BIC and CAIC) for Gauge Length data

Model	MLE's	AIC	BIC	CAIC
Rayleigh Distribution	$\hat{\beta}$ =0.3204	327.5	329.6	327.5
Lomax Distribution	$\hat{\lambda}$ =0.0063 $\hat{\theta}$ =51.455	272.1	276.4	272.3
Exponential Distribution	$\hat{\lambda}$ =0.3269	268.9	271.0	268.9
Exponential Lomax Distribution	$\hat{\alpha}$ =7.47 $\hat{\beta}$ =2.664 $\hat{\lambda}$ =0.002	140.0	146.4	140.4
New Weibull Lomax Distribution	$\hat{\beta}$ =5.125 $\hat{\sigma}$ =11.395 $\hat{K}$ =0.003	139.7	146.2	140.1
Weighted Exponential-Lomax (WEL)	$\hat{\alpha}$ = **5.45547** $\hat{\lambda}$ = **0.00884** $\hat{\beta}$ = **1.901**	**129.053**	**128.451**	**129.459**

6. Information of Inequality and Uncertainty

Entropy is used to measure the variation of uncertainty for the continuous random variable X. In this section, following entropies are discussed.

Renyi entropy for the WEL distribution is calculated as follows:

$$\gamma_v\{f(x)\} = \frac{1}{1-v} \times ln \int_{-\infty}^{\infty} f^v(x)dx$$

where $v > 0$ and $v \neq 1$

$$\gamma_v\{f(x)\} = \frac{1}{1-v} \times ln\left[\left(\frac{\beta}{\alpha}\right)^{1-v} \frac{1}{v^{2v-\frac{v}{\alpha}+\frac{1}{\alpha}\lambda^{\frac{-v}{\lambda}+\frac{1}{\alpha}}}} \sqrt{\left[2v - \frac{1}{\alpha}(1-v)\right]}\right] \tag{32}$$

Q-Entropy for WEL distribution is obtained as

$$H_q(x) = \frac{1}{q-1} \times \ln\left[1 - \left[\left(\frac{\beta}{\alpha}\right)^{1-q} \frac{1}{v^{2q-\frac{q}{\alpha}+\frac{1}{\alpha}\lambda^{\frac{-q}{\lambda}+\frac{1}{\alpha}}}} \sqrt{\left[2q - \frac{1}{\alpha}(1-q)\right]}\right]\right],\ q > 0, q \neq 1 \tag{33}$$

Shannon entropy [21] is the special case of Renyi entropy when $v \to 1$. Shannon entropy for WEL distribution is

$$-H(x) = \int_{-\beta}^{\infty} f(x) ln\, f(x)dx$$

$$-H(x) = 2ln\lambda + ln\alpha - ln\beta + \frac{2\alpha-1}{\alpha}[(1-\gamma) - (ln\boldsymbol{\lambda})] - 2 \tag{34}$$

where γ is the Euler Mascherroni constant.

Theorem 6.1: If x follows the WEL distribution with pdf $f(x)$ and cdf $F(x)$ then the Lorenz and Bonferroni curves given as

$$L(x) = \frac{\left[\left(\frac{1}{\lambda^{1/\alpha}}\sqrt{\frac{1}{\alpha}+2}\,\gamma_{\frac{1}{\alpha}+2}\left(\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right)\right) - \left(\gamma_2\left(\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right)\right)\right]}{\beta\left[\frac{1}{\lambda^{1/\alpha}}\sqrt{\left[\frac{1}{\alpha}+2\right]}-1\right]^2} \tag{35}$$

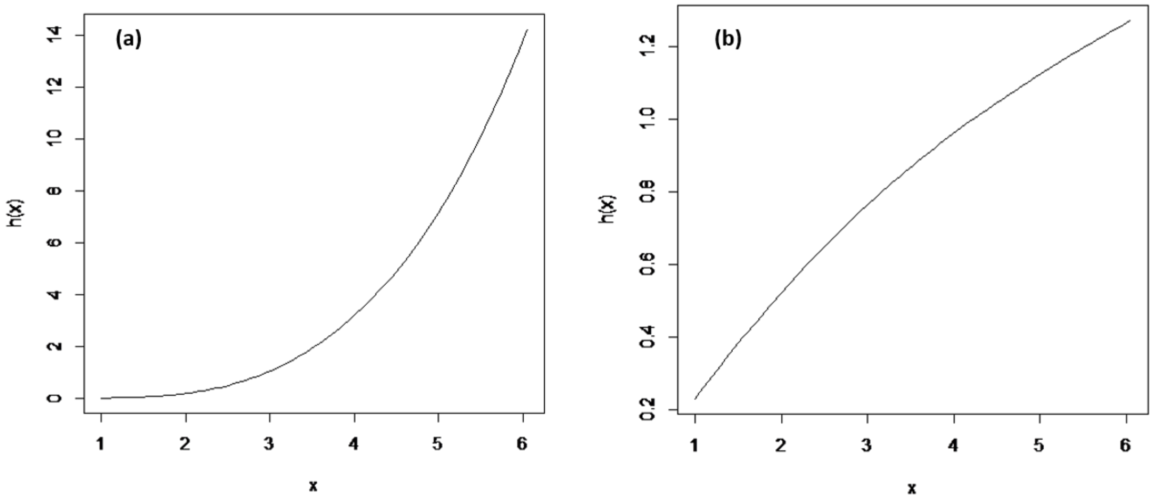

Fig. 8. Hazard rate graphs for the Gauge length data in (a) and Failure Time of 84 Aircraft Windshield data in (b)

Table 2. MLEs and the Statistics (AIC, BIC and CAIC) for Failure Time of 84 Aircraft Windshield data.

Model	MLE's	AIC	BIC	CAIC
Lomax	$\hat{\lambda}$ =131789. 78 $\hat{\theta}$=51425.35	333.9767	338.8620	334.1230
Exponential Lomax Distribution	$\hat{\alpha}$=20074.5 097 $\hat{\beta}$ =26257.6 808 $\hat{\lambda}$ =3.62 61	288.7994	296.1273	289.0957
Beta Lomax distribution	$\hat{\alpha}$=4.8307 $\hat{\beta}$ =118.8374 $\hat{\theta}$=3.6036 $\hat{\lambda}$ =33.6387	285.4354	295.2060	285.935
Gamma Lomax distribution	$\hat{\alpha}$= 52001.4 994 $\hat{\beta}$ =37029.6 583 $\hat{\lambda}$ =3.5876	282.8083	290.1363	283.1046
Weighted Exponential-Lomax (WEL)	$\hat{\alpha}$= **1.4279** $\hat{\lambda}$ = **0.00487** $\hat{\beta}$ **0.040**	**274.039**	**273.811**	**274.339**

The required numerical calculations are done by using the Package R software. The model selection is carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion) and the CAIC (consistent Akaike information criteria):

$$B(x)=\frac{\left(\frac{1}{\lambda^{1/\alpha}}\sqrt{\frac{1}{\alpha}+2}\,\gamma_{\frac{1}{\alpha}+2}\left(\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right)\right)-\left(\gamma_2\left(\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right)\right)}{\left(\beta\left[\frac{1}{\lambda^{1/\alpha}}\sqrt{\left[\frac{1}{\alpha}+2\right]}-1\right]^2\left(1-e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}\left[1+\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}\right]\right)\right)} \quad (36)$$

Proof: Let the cdf and pdf of WEL distribution in equations (5) and (6), the Lorenz and Bonferroni curves for a WEL random variable X

$$L(x)=\int_{-\infty}^{x}\frac{tf(t)dt}{\mu} \quad (37)$$

$$B(x)=\frac{L(x)}{F(x)} \quad (38)$$

Where μ is mean of the WEL distribution.

$$L(x)=\frac{1}{\mu\beta}\int_{-\beta}^{x} x\frac{\lambda^2\alpha}{\beta}\left[\frac{\beta}{x+\beta}\right]^{-2\alpha+1} e^{-\lambda\left[\frac{\beta}{x+\beta}\right]^{-\alpha}}dx$$

Simplifying it we get the expression in (35) and using it in equation (38) we get the expression in equation (36).

7. Maximum Likelihood Estimation (MLE)

In this section, parameters of the WEL distribution are estimated by Maximum likelihood estimation method. Let $x_1, x_2, \ldots\ldots, x_n$ be a random sample of size n from WEL distribution. Then the Likelihood function (L) of equation (6) is given by

$$L(\alpha,\lambda,\beta) = \frac{\lambda^{2n}\alpha^n}{\beta^n}\prod_{i=1}^{n}\left[\frac{\beta}{x_i+\beta}\right]^{-2\alpha+1} e\sum_{i=1}^{n}\left[\frac{\beta}{x_i+\beta}\right]^{-\alpha^{-\lambda}} \tag{39}$$

Hence the log-likelihood function, $l = \ln$ L becomes

$$l(\alpha,\lambda,\beta) = 2n\ln\lambda + n\ln\alpha - n\ln\beta + 2\alpha\sum_{i=1}^{n}\ln\left[\frac{\beta}{x_i+\beta}\right] - \sum_{i=1}^{n}\ln\left[\frac{\beta}{x_i+\beta}\right] - \lambda\sum_{i=1}^{n}\left[\frac{\beta}{x_i+\beta}\right]^{-\alpha} \tag{40}$$

Differentiating equation (40) with respect to λ and α respectively, we get

$$\frac{dl(\alpha,\lambda,\beta)}{d\lambda} = 2n\frac{1}{\lambda} - \sum_{i=1}^{n}\left[\frac{\beta}{x_i+\beta}\right]^{-\alpha} \tag{41}$$

$$\frac{dl(\alpha,\lambda,\beta)}{d\alpha} = \frac{n}{\alpha} + 2\sum_{i=1}^{n}\ln\left[\frac{\beta}{x_i+\beta}\right] - \lambda\frac{d}{d\alpha}\sum_{i=1}^{n}\left[\frac{\beta}{x_i+\beta}\right]^{-\alpha} \tag{42}$$

8. Applications

Data I: The data set (gauge length of 10mm) from Kundu and Raqab.[24] The data set holds sixty-three observations as:
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020
Data II: Following data is reported in the book "Weibull Models" by Murthy et al.[25] (page 297). The failure times of 84 Aircraft Windshield is 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

9. Conclusions

In this article, a new three parameter distribution is derived named as Weighted-Exponential Lomax (WEL) distribution. Various statistical properties including cdf, moments, median, mode, geometric mean, harmonic mean, mean deviation, reliability measures (reliability function, hazard function, reversed hazard function, cumulative hazard function and mean residual function), entropies (Renny, Q and Shannon), Lorenz and Bonferroni curves, quantile function and random number generator of the proposed model have been derived. The graphs of the necessary function have been provided. From the hazard rate graphs, it can be seen that the distribution having IFR. The parameters of the WEL distribution are estimated by MLE and the WEL model is applied on two data set (Gauge length time and Failure Time of 84 Aircraft Windshield (Fig. 7)). From table 1 & 2, it is observed that the WEL distribution gives the smaller values of the AIC, BIC & CAIC as compare to some other well-known models so more flexibility of the new distribution is found as compare to the previously existing probability models. From Fig. 8, it can be seen that the hazard function of the WEL distribution for gauge length and failure time of 84 Aircraft Windshield is IFR.

Conflicts of Interest

The authors declare no conflict of interest.

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