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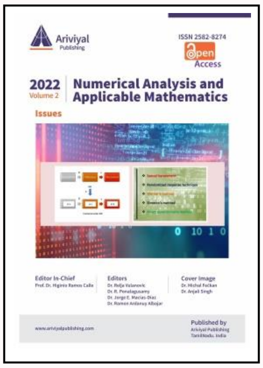

The Approximate Solution of First Order Delay Differential Equations Using Extended Third Derivative Block Backward Differentiation Formulae

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Abstract: The formulation of extended third derivative block backward differentiation formulae was presented for the solution of first order delay differential equations (DDEs) without the application of interpolation techniques in computing the delay term. The delay term was computed by a valid expression of sequence. By matrix transposition procedure, the discrete schemes of the proposed method were carried-out through its continuous derivations with the help of linear multistep collocation approach. The convergence and stability analysis of the method were satisfied. The performances of the proposed method on numerical experiments of some first order DDEs revealed that the scheme for step number k = 4 performed better and faster in terms of efficiency, accuracy, consistency, convergence, region of absolute stability and Central Processing Unit Time (CPUT) at fixed step size than the schemes for step numbers k = 3 and 2 when compared with their exact solutions and other existing methods.

Keywords: First order delay differential equations; extended third derivative backward differentiation formulae; block method

1. Introduction

So many scholars have studied and revealed that delay differential equations can be applied in every real life situations such as engineering, physics, medicine and economics which proved more realistic as demonstrated by Tziperman et al., Ballen et al.[1,2] In accordance with reality, ignoring delays in real life situations means ignoring reality because the solution of delay differential equations takes into consideration the current state and the history part of the system being modeled.

One of the setbacks experienced by these researchers in the use of interpolation techniques to calculate the delay term of DDEs which was studied by Majid Z.A. et al.,[3] that the computational method use in solving DDEs should be the same with the interpolating polynomials which is very hard to achieve; otherwise, the accuracy of the method will not be preserved. Therefore, it is required that in the evaluation of the delay term, using an accurate and efficient formula shall be considered.

In order to overcome the limitation posed by using interpolation techniques in checking the delay term, we shall apply the valid expression of the sequence formulated by Sirisena et al.,[4] and incorporate it into the first order delay differential equations before its evaluation. This approach has been successfully applied by Osu B.O. & Chibusi C. et al.,[5,6] in finding the numerical solution of first order delay differential equations.

In this paper, we shall formulate and apply extended third derivative block backward differentiation formulae to solve some first order delay differential equations (DDEs) as developed by Bellman R. et al.,[7]

$$y'(t) = f\big(t, y(t), y(t-\tau)\big), \text{ for } t > t_0, \tau > 0$$

$$y(t) = u(t), \text{ for } t \le t_0 \qquad (1)$$

where $u(t)$is the initial function, τ is called the delay, $t-\tau$ is called the delay argument and $y(t-\tau)$ is the solution of the delay term. The results obtained after the application of the proposed method shall be compared to other existing methods studied by[5,6] to prove its advantage.

2. Construction Approach

2.1. Construction of Extended Third Derivative Backward Differentiation Formulae Method

Osu et al.,[5] applied a basis function to develop the continuous third derivative backward differentiation formula through the k-step generalized Backward Differentiation Formulae Methods formulated by[2] which can be expressed as

$$y_{v+p} = \sum_{a=0}^{p} \alpha_a(x) y_{v+a} + q\beta_a(x) f(x_a, y(x_a)) + q^2\gamma_a(x) g(x_a, y(x_a)) + q^3\delta_a(x) m(x_a, y(x_a)) \quad (2)$$

Then from (3), the continuous extended third derivative backward differentiation formula is obtain as,

$$y_{v+p} = \sum_{a=0}^{p} \alpha_a(x) y_{v+a} + q\beta_a(x) f\big(x_a, y(x_a)\big) + q^2\gamma_a(x) g\big(x_a, y(x_a)\big) + q^3\delta_a(x) m\big(x_a, y(x_a)\big) + q^4\eta_a(x) w(x_a, y(x_a)) \quad (3)$$

where $y_{v+a} = y(x_v + aq), f_{v+p} \equiv f(x_v + aq), y(x_v + aq), y'(x_v + aq),$

$g_{v+p} \equiv \frac{df}{dx}\Big|\begin{matrix} x = x_{v+p} \\ y = y_{v+p} \end{matrix}, m_{v+p} \equiv \frac{d^2f}{dx^2}\Big|\begin{matrix} x = x_{v+p} \\ y = y_{v+p} \end{matrix}, W_{v+p} \equiv \frac{d^2f}{dx^2}\Big|\begin{matrix} x = x_{v+p} \\ y = y_{v+p} \end{matrix}$ and $\alpha_a(x), \beta_a(x), \gamma_a(x) and \delta_a(x)$

are continuous coefficients of the method defined as

$$\alpha_a(x) = \sum_{s=0}^{j+z-1} \alpha_{a,s+1} x^s \text{ for a=\{0,1,...,j-1\}} \quad (4)$$

$$q\beta_z(x) = \sum_{s=0}^{j+z-1} q\beta_{a,s+1} x^s \text{ for a=\{0,1,...,z-1\}} \quad (5)$$

$$q^2\gamma_a(x) = \sum_{s=1}^{j+z-1} q^2\gamma_{a,s+1} x^s \text{ for a=\{0,1,...,z-1\}} \quad (6)$$

$$q^3\delta_a(x) = \sum_{s=1}^{j+z-1} q^3\delta_{a,s+1} x^s \text{ for a=\{0,1,...,z-1\}} \quad (7)$$

$$q^4\delta_a(x) = \sum_{s=1}^{j+z-1} q^4 w_{a,s+1} x^s \text{ for a=\{0,1,...,z-1\}} \quad (8)$$

Where $X_0, \dots, X_{z-1}$ are the Z collocation points, X_{v+a}, a={0,1,2,...,j-1} are the j arbitrarily chosen interpolation points, q is the constant step size and k is the step number of the proposed method. By matrix transposition procedure applied by Osu et al.,[5] the discrete schemes of the proposed method for step numbers k = 2, 3 and 4 were formulated through its continuous derivations with the help of linear multistep collocation approach by matrix inversion process.

2.2. Construction of Extended Third Derivative Block Backward Differentiation Formulae Method for k = 2

For this case, $j = 2$ and $z = 4$. Hence, (3) takes the form

$$y(x) = \alpha_0(x) y_v + \alpha_1(x) y_{v+1} + q\beta_2(x) f_{v+2} + q^2\gamma_2(x) g_{v+2} + q^3\delta_2(x) m_{v+2} + q^3\eta_3(x) w_{v+3} \quad (9)$$

By simplifying and evaluating the inverse of the matrix equation formed by Osu et al.[5]
At $x = x_{v+3}, x = x_{v+2}$ and its derivatives at $x = x_{v+1}$ using Maple 18, the following discrete schemes were obtained

$$y_{v+1} = -\frac{451}{360} q^3 m_{v+2} + \frac{49}{360} q^3 w_{v+3} + \frac{46}{15} q^2 g_{v+2} + \frac{137}{30} q f_{v+1} - \frac{107}{30} q f_{v+2} + y_v$$

$$y_{v+2} = -\frac{7}{137} y_v + \frac{144}{137} y_{v+1} + \frac{130}{137} q f_{v+2} - \frac{58}{137} q^2 g_{v+2} + \frac{16}{137} q^3 m_{v+2} - \frac{4}{411} q^3 w_{v+3}$$

$$y_{v+3} = -\frac{4}{137} y_v + \frac{141}{137} y_{v+1} + \frac{270}{137} q f_{v+2} + \frac{6}{137} q^2 g_{v+2} + \frac{77}{274} q^3 m_{v+2} + \frac{5}{274} q^3 w_{v+3} \quad (10)$$

2.3. Construction of Extended Third Derivative Block Backward Differentiation Formulae Method for k = 3

Let $j = 3$ and $z = 4$. Hence, (3) takes the form

$$y(x) = \alpha_0(x) y_v + \alpha_1(x) y_{v+1} + \alpha_2(x) y_{v+2} + q\beta_3(x) f_{v+3} + q^2\gamma_3(x) g_{v+3} + q^3\delta_3(x) m_{v+3} + q^3\eta_4(x) w_{v+4} \quad (11)$$

Simplifying and evaluating the inverse of the matrix equation formed by [5] at $x = x_{v+4}, x = x_{v+3}$ and its derivative at $x = x_{v+2}$ and $x = x_{v+1}$ using Maple 18, we obtained

$$y_{v+1} = -\frac{358}{1797}q^3m_{v+3} + \frac{239}{14376}q^3w_{v+4} + \frac{2301}{4792}q^2g_{v+3} - \frac{6943}{9584}qf_{v+1} - \frac{4697}{9584}qf_{v+3} - \frac{257}{2396}y_v + \frac{2653}{2396}y_{v+2}$$

$$y_{v+2} = -\frac{2501}{5634}q^3m_{v+3} + \frac{1697}{61974}q^3w_{v+4} + \frac{4742}{3443}q^2g_{v+3} + \frac{27772}{10329}qf_{v+2} - \frac{6074}{3443}qf_{v+3} - \frac{779}{10329}y_v + \frac{11108}{10329}y_{v+1}$$

$$y_{v+3} = \frac{58}{6943}y_v - \frac{729}{6943}y_{v+1} + \frac{7614}{6943}y_{v+2} + \frac{6330}{6943}qf_{v+3} - \frac{2610}{6943}q^2g_{v+3} + \frac{585}{6943}q^3m_{v+3} - \frac{27}{6943}q^3w_{v+4}$$

$$y_{v+4} = -\frac{6}{6943}y_v - \frac{164}{6943}y_{v+1} + \frac{7113}{6943}y_{v+2} + \frac{13710}{6943}qf_{v+3} + \frac{270}{6943}q^2g_{v+3} + \frac{3949}{13886}q^3m_{v+3} + \frac{245}{13886}q^3w_{v+4} \tag{12}$$

2.4. Construction of Extended Third Derivative Block Backward Differentiation Formulae Method for k = 4

With the same procedure, let $j = 4$ and $z = 4$. Hence, (3) takes the form

$$y(x) = \alpha_0(x)y_v + \alpha_1(x)y_{v+1} + \alpha_2(x)y_{v+2} + \alpha_3(x)y_{v+3} + q\beta_4(x)f_{v+4} + q^2\gamma_4(x)g_{v+4} + q^3\delta_4(x)m_{v+4} + q^3\eta_5(x)w_{v+5} \tag{13}$$

Simplifying and evaluating the inverse of the matrix equation formed by [5] at $x = x_{v+5}, x = x_{v+4}$ and its derivative at $x = x_{v+3}, x = x_{v+2}$ and $x = x_{v+1}$ using Maple 18, the following discrete schemes are obtained

$$y_{v+1} = -\frac{248359}{1207491}q^3m_{v+4} - \frac{15938}{1207491}q^3w_{v+5} - \frac{3874571}{7244946}q^2g_{v+4} - \frac{6933226}{10867419}qf_{v+1} + \frac{11945267}{21734838}qf_{v+4} - \frac{396965}{4829964}y_v + \frac{3474631}{1609988}y_{v+2} - \frac{1299241}{1207491}y_{v+3}$$

$$y_{v+2} = -\frac{696541}{4730667}q^3m_{v+4} + \frac{37804}{4730667}q^3w_{v+5} + \frac{673941}{1576889}q^2g_{v+4} - \frac{3466613}{3153778}qf_{v+2} - \frac{1510807}{3153778}qf_{v+4} + \frac{32281}{1576889}y_v - \frac{504332}{1576889}y_{v+1} + \frac{2048940}{1576889}y_{v+3}$$

$$y_{v+3} = -\frac{2400291}{10274209}q^3m_{v+4} + \frac{96918}{10274209}q^3w_{v+5} + \frac{17292573}{20548418}q^2g_{v+4} + \frac{20799678}{10274209}qf_{v+3} - \frac{23766471}{20548418}qf_{v+4} + \frac{545977}{41096836}y_v - \frac{1630755}{10274209}y_{v+1} + \frac{47073879}{41096836}y_{v+2}$$

$$y_{v+4} = -\frac{7803}{3466613}y_v + \frac{88832}{3466613}y_{v+1} - \frac{561168}{3466613}y_{v+2} + \frac{3946752}{3466613}y_{v+3} + \frac{3059700}{3466613}qf_{v+4} - \frac{1188360}{3466613}q^2g_{v+4} + \frac{232992}{3466613}q^3m_{v+4} - \frac{6912}{3466613}q^3w_{v+5}$$

$$y_{v+5} = \frac{15261}{6933226}y_v - \frac{61545}{3466613}y_{v+1} + \frac{221875}{6933226}y_{v+2} + \frac{3409590}{3466613}y_{v+3} + \frac{6943965}{3466613}qf_{v+4} + \frac{22545}{3466613}q^2g_{v+4} + \frac{1043650}{3466613}q^3m_{v+4} + \frac{54740}{3466613}q^3w_{v+5} \tag{14}$$

3. Convergence analysis

In this section, the order, error constant, consistency, zero stability and region of the absolute stability of (10), (12) and (14) are analyzed.

3.1. Order and Error Constant

The ETDBBDFM (3) is said to be of order e if $C_0 = C_1 = \cdots C_e = 0$ and the first non-zero coefficient $C_{e+1} \neq 0$ is the error constant as developed by Lambert J. D. 1973.[8] The order and error constant for (10) are obtained as follows

$C_0 = C_1 = (0 \quad 0 \quad 0)^T$ but $C_2 = \left(\frac{46}{15} - \frac{58}{137}\ \frac{6}{137}\right)^T$. Therefore, (10) has order $e = 1$ and error constants, $\left(\frac{46}{15} - \frac{58}{137}\ \frac{6}{137}\right)^T$.

With the same approach, (12) can be presented as

$C_0 = C_1 = (0 \quad 0 \quad 0 \quad 0)^T$ but $C_2 = \left(\frac{2301}{4792}\ \ \frac{4742}{3443} - \frac{2610}{6943}\ \ \frac{270}{6943}\right)^T$. Therefore, (12) has order $e = 1$ and error constants $\left(\frac{2301}{4792}\ \ \frac{4742}{3443} - \frac{2610}{6943}\ \ \frac{270}{6943}\right)^T$.

Applying the same approach, (14) can be obtained as

$C_0 = C_1 = (0 \quad 0 \quad 0 \quad 0 \quad 0)^T$ but $C_2 = \left(-\frac{3874571}{7244946}\ \ \frac{673941}{1576889}\ \ \frac{17292573}{20548418} - \frac{1188360}{3466613} - \frac{86620235}{6933226}\right)^T$ Therefore, (14) has order $e = 1$ and error constants $\left(-\frac{3874571}{7244946}\ \ \frac{673941}{1576889}\ \ \frac{17292573}{20548418} - \frac{1188360}{3466613} - \frac{86620235}{6933226}\right)^T$.

3.2. Consistency

Since $e = 1$ in (10), (12) and (14) satisfying the condition for consistency of order $e \geq 1$ as stated by Lambert J. D. 1973,[8] then the discrete schemes are said to be consistent.

3.3. Zero Stability

The discrete schemes (10), (12) and (14) are said to be zero stable if the no root of the first characteristic polynomial is greater than 1. The zero stability for (10) is analyzed as follows

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{144}{137} & 1 & 0 \\ -\frac{141}{137} & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{7}{137} \\ 0 & 0 & \frac{4}{137} \end{pmatrix}\begin{pmatrix} y_{v-2} \\ y_{v-2} \\ y_v \end{pmatrix}$$

$$+q\begin{pmatrix} -\frac{137}{30} & -\frac{107}{30} & 0 \\ 0 & \frac{130}{137} & 0 \\ 0 & \frac{270}{137} & 0 \end{pmatrix}\begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \end{pmatrix} + q\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{v-2} \\ f_{v-1} \\ f_v \end{pmatrix} + q^2\begin{pmatrix} 0 & \frac{46}{15} & 0 \\ 0 & -\frac{58}{137} & 0 \\ 0 & \frac{6}{137} & 0 \end{pmatrix}\begin{pmatrix} g_{v+1} \\ g_{v+2} \\ g_{v+3} \end{pmatrix} + q^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{v-2} \\ g_{v-1} \\ g_v \end{pmatrix}$$

$$+q^3\begin{pmatrix} 0 & -\frac{451}{360} & 0 \\ 0 & \frac{16}{137} & 0 \\ 0 & \frac{77}{274} & 0 \end{pmatrix}\begin{pmatrix} m_{v+1} \\ m_{v+2} \\ m_{v+3} \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} m_{v-2} \\ m_{v-1} \\ m_v \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & \frac{49}{360} \\ 0 & 0 & -\frac{4}{411} \\ 0 & 0 & \frac{5}{274} \end{pmatrix}\begin{pmatrix} w_{v+1} \\ w_{v+2} \\ w_{v+3} \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w_{v-2} \\ w_{v-1} \\ w_v \end{pmatrix}$$

Where

$D_2^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{144}{137} & 1 & 0 \\ -\frac{141}{137} & 0 & 1 \end{pmatrix}, D_1^{(1)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{7}{137} \\ 0 & 0 & \frac{4}{137} \end{pmatrix}, U_2^{(1)} = \begin{pmatrix} -\frac{137}{30} & -\frac{107}{30} & 0 \\ 0 & \frac{130}{137} & 0 \\ 0 & \frac{270}{137} & 0 \end{pmatrix}, H_2^{(1)} = \begin{pmatrix} 0 & \frac{46}{15} & 0 \\ 0 & -\frac{58}{137} & 0 \\ 0 & \frac{6}{137} & 0 \end{pmatrix}, C_2^{(1)} = \begin{pmatrix} 0 & -\frac{451}{360} & 0 \\ 0 & -\frac{16}{137} & 0 \\ 0 & \frac{77}{274} & 0 \end{pmatrix}$ and

$B_2^{(1)} = \begin{pmatrix} 0 & 0 & \frac{49}{360} \\ 0 & 0 & -\frac{4}{411} \\ 0 & 0 & \frac{5}{274} \end{pmatrix}.$

The first characteristic polynomial is presented as

$$\mu(\varepsilon) = \det\left(\varepsilon D_2^{(1)} - D_1^{(1)}\right) = \left|\varepsilon D_2^{(1)} - D_1^{(1)}\right| = 0. \tag{15}$$

Now we have,

$$\mu(\varepsilon) = \left|\varepsilon\begin{pmatrix} 1 & 0 & 0 \\ -\frac{144}{137} & 1 & 0 \\ -\frac{141}{137} & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{7}{137} \\ 0 & 0 & \frac{4}{137} \end{pmatrix}\right| = \left|\begin{pmatrix} \varepsilon & 0 & 0 \\ -\frac{144}{137}\varepsilon & \varepsilon & 0 \\ -\frac{141}{137}\varepsilon & 0 & \varepsilon \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \frac{7}{137} \\ 0 & 0 & \frac{4}{137} \end{pmatrix}\right| \Rightarrow \mu(\varepsilon) = \begin{pmatrix} \varepsilon & 0 & 1 \\ -\frac{144}{137}\varepsilon & \varepsilon & -\frac{7}{137} \\ -\frac{141}{137}\varepsilon & 0 & \varepsilon - \frac{4}{137} \end{pmatrix}$$

Using Maple (18) software,

$\mu(\varepsilon) = \varepsilon^3 + \varepsilon^2$

$\Rightarrow \varepsilon^3 + \varepsilon^2 = 0$

$\Rightarrow \varepsilon_1 = -1, \varepsilon_2 = 0, \varepsilon_3 = 0$. Since $|\varepsilon_i| < 1, i = 1,2,3$, (10) is zero stable.

By the same procedure for (12)

$$\begin{pmatrix} 1 & -\frac{2653}{2396} & 0 & 0 \\ \frac{11108}{10329} & 1 & 0 & 0 \\ \frac{729}{6943} & -\frac{7614}{6943} & 1 & 0 \\ \frac{164}{6943} & -\frac{7113}{6943} & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{257}{2396} \\ 0 & 0 & 0 & \frac{779}{10329} \\ 0 & 0 & 0 & -\frac{58}{6943} \\ 0 & 0 & 0 & \frac{6}{6943} \end{pmatrix}\begin{pmatrix} y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_v \end{pmatrix}$$

$$+q\begin{pmatrix} -\frac{6943}{9584} & 0 & -\frac{4697}{9584} & 0 \\ 0 & \frac{27772}{10329} & -\frac{6074}{3443} & 0 \\ 0 & 0 & \frac{6330}{6943} & 0 \\ 0 & 0 & \frac{13710}{6943} & 0 \end{pmatrix}\begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+4} \end{pmatrix} + q\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_{v} \end{pmatrix} + q^2\begin{pmatrix} 0 & 0 & \frac{2301}{4792} & 0 \\ 0 & 0 & \frac{4742}{3443} & 0 \\ 0 & 0 & -\frac{2610}{6943} & 0 \\ 0 & 0 & \frac{270}{6943} & 0 \end{pmatrix}\begin{pmatrix} g_{v+1} \\ g_{v+2} \\ g_{v+3} \\ g_{v+4} \end{pmatrix} + q^2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{v-3} \\ g_{v-2} \\ g_{v-1} \\ g_{v} \end{pmatrix}$$

$$+q^3\begin{pmatrix} 0 & 0 & -\frac{358}{1797} & 0 \\ 0 & 0 & -\frac{2501}{5634} & 0 \\ 0 & 0 & \frac{585}{6943} & 0 \\ 0 & 0 & \frac{3949}{13886} & 0 \end{pmatrix}\begin{pmatrix} m_{v+1} \\ m_{v+2} \\ m_{v+3} \\ m_{v+4} \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} m_{v-3} \\ m_{v-2} \\ m_{v-1} \\ m_{v} \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & \frac{239}{14376} & 0 \\ 0 & 0 & \frac{1697}{61974} & 0 \\ 0 & 0 & -\frac{27}{6943} & 0 \\ 0 & 0 & \frac{245}{13886} & 0 \end{pmatrix}\begin{pmatrix} w_{v+1} \\ w_{v+2} \\ w_{v+3} \\ w_{v+4} \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w_{v-3} \\ w_{v-2} \\ w_{v-1} \\ w_{v} \end{pmatrix}$$

where

$$D_2^{(2)} = \begin{pmatrix} 1 & -\frac{2653}{2396} & 0 & 0 \\ \frac{11108}{10329} & 1 & 0 & 0 \\ \frac{729}{6943} & -\frac{7614}{6943} & 1 & 0 \\ \frac{164}{6943} & -\frac{7113}{6943} & 0 & 1 \end{pmatrix}, D_1^{(2)} = \begin{pmatrix} 0 & 0 & 0 & \frac{257}{2396} \\ 0 & 0 & 0 & \frac{779}{10329} \\ 0 & 0 & 0 & -\frac{58}{6943} \\ 0 & 0 & 0 & \frac{6}{6943} \end{pmatrix}, U_2^{(2)} = \begin{pmatrix} -\frac{6943}{9584} & 0 & -\frac{4697}{9584} & 0 \\ 0 & \frac{27772}{10329} & -\frac{6074}{3443} & 0 \\ 0 & 0 & \frac{6330}{6943} & 0 \\ 0 & 0 & \frac{13710}{6943} & 0 \end{pmatrix}, H_2^{(2)} = \begin{pmatrix} 0 & 0 & \frac{2301}{4792} & 0 \\ 0 & 0 & \frac{4742}{3443} & 0 \\ 0 & 0 & -\frac{2610}{6943} & 0 \\ 0 & 0 & \frac{270}{6943} & 0 \end{pmatrix},$$

$$C_2^{(2)} = \begin{pmatrix} 0 & 0 & -\frac{358}{1797} & 0 \\ 0 & 0 & -\frac{2501}{5634} & 0 \\ 0 & 0 & \frac{585}{6943} & 0 \\ 0 & 0 & \frac{3949}{13886} & 0 \end{pmatrix} \text{ and } B_2^{(2)} = \begin{pmatrix} 0 & 0 & \frac{239}{14376} & 0 \\ 0 & 0 & \frac{1697}{61974} & 0 \\ 0 & 0 & -\frac{27}{6943} & 0 \\ 0 & 0 & \frac{245}{13886} & 0 \end{pmatrix}$$

The first characteristic polynomial is presented as

$$\mu(\varepsilon) = \det\left(\varepsilon D_2^{(2)} - D_1^{(2)}\right) = \left|\varepsilon D_2^{(2)} - D_1^{(2)}\right| = 0. \quad (16)$$

Now we have,

$$\mu(\varepsilon) = \left|\varepsilon\begin{pmatrix} 1 & -\frac{2653}{2396} & 0 & 0 \\ \frac{11108}{10329} & 1 & 0 & 0 \\ \frac{729}{6943} & -\frac{7614}{6943} & 1 & 0 \\ \frac{164}{6943} & -\frac{7113}{6943} & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \frac{257}{2396} \\ 0 & 0 & 0 & \frac{779}{10329} \\ 0 & 0 & 0 & -\frac{58}{6943} \\ 0 & 0 & 0 & \frac{6}{6943} \end{pmatrix}\right| = \left|\begin{pmatrix} \varepsilon & -\frac{2653}{2396}\varepsilon & 0 & 0 \\ \frac{11108}{10329}\varepsilon & \varepsilon & 0 & 0 \\ \frac{729}{6943}\varepsilon & -\frac{7614}{6943}\varepsilon & \varepsilon & 0 \\ \frac{164}{6943}\varepsilon & -\frac{7113}{6943}\varepsilon & 0 & \varepsilon \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \frac{257}{2396} \\ 0 & 0 & 0 & \frac{779}{10329} \\ 0 & 0 & 0 & -\frac{58}{6943} \\ 0 & 0 & 0 & \frac{6}{6943} \end{pmatrix}\right|$$

$$\Rightarrow \mu(\varepsilon) = \begin{pmatrix} \varepsilon & -\frac{2653}{2396}\varepsilon & 0 & -\frac{257}{2396} \\ \frac{11108}{10329}\varepsilon & \varepsilon & 0 & -\frac{779}{10329} \\ \frac{729}{6943}\varepsilon & -\frac{7614}{6943}\varepsilon & \varepsilon & \frac{58}{6943} \\ \frac{164}{6943}\varepsilon & -\frac{7113}{6943}\varepsilon & 0 & -\frac{6}{6943} \end{pmatrix}$$

Applying Maple (18) software, the following are obtained

$$\mu(\varepsilon) = -\frac{1180310}{6187071}\varepsilon^4 - \frac{1180310}{6187071}\varepsilon^3$$

$$\Rightarrow -\frac{1180310}{6187071}\varepsilon^4 - \frac{1180310}{6187071}\varepsilon^3 = 0$$

$\Rightarrow \varepsilon_1 = -1, \varepsilon_2 = 0, \varepsilon_3 = 0, \varepsilon_4 = 0$. Since $|\varepsilon_i| < 1, i = 1,2,3,4$, thus (12) is zero stable.

Applying the same procedure for (14)

$$\begin{pmatrix} 1 & -\frac{3474631}{1609988} & \frac{1299241}{1207491} & 0 & 0 \\ \frac{504332}{1576889} & 1 & -\frac{2048940}{1576889} & 0 & 0 \\ \frac{1630755}{10274209} & -\frac{47073879}{41096836} & 1 & 0 & 0 \\ -\frac{88832}{3466613} & \frac{561168}{3466613} & -\frac{3946752}{3466613} & 1 & 0 \\ -\frac{61545}{3466613} & -\frac{221875}{6933226} & -\frac{3409590}{3466613} & 0 & 1 \end{pmatrix}\begin{pmatrix} y_{v+1} \\ y_{v+2} \\ y_{v+3} \\ y_{v+4} \\ y_{v+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{396965}{4829964} \\ 0 & 0 & 0 & 0 & -\frac{32281}{1576889} \\ 0 & 0 & 0 & 0 & -\frac{545977}{41096836} \\ 0 & 0 & 0 & 0 & \frac{7803}{3466613} \\ 0 & 0 & 0 & 0 & -\frac{15261}{6933226} \end{pmatrix}\begin{pmatrix} y_{v-4} \\ y_{v-3} \\ y_{v-2} \\ y_{v-1} \\ y_{v} \end{pmatrix}$$

$$q\begin{pmatrix} -\frac{6933226}{10867419} & 0 & 0 & \frac{11945267}{21734838} & 0 \\ 0 & \frac{3466613}{3153778} & 0 & \frac{1510807}{3153778} & 0 \\ 0 & 0 & \frac{20799678}{10274209} & -\frac{23766471}{20548418} & 0 \\ 0 & 0 & 0 & \frac{3059700}{3466613} & 0 \\ 0 & 0 & 0 & \frac{6943965}{3466613} & 0 \end{pmatrix}\begin{pmatrix} f_{v+1} \\ f_{v+2} \\ f_{v+3} \\ f_{v+4} \\ f_{v+5} \end{pmatrix} + q\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{v-4} \\ f_{v-3} \\ f_{v-2} \\ f_{v-1} \\ f_{v} \end{pmatrix} + q^2\begin{pmatrix} 0 & 0 & 0 & \frac{3874571}{7244946} & 0 \\ 0 & 0 & 0 & \frac{673941}{1576889} & 0 \\ 0 & 0 & 0 & \frac{17292573}{20548418} & 0 \\ 0 & 0 & 0 & -\frac{1188360}{3466613} & 0 \\ 0 & 0 & 0 & \frac{22545}{3466613} & 0 \end{pmatrix}\begin{pmatrix} g_{v+1} \\ g_{v+2} \\ g_{v+3} \\ g_{v+4} \\ g_{v+5} \end{pmatrix} +$$

$$q^2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{v-4} \\ g_{v-3} \\ g_{v-2} \\ g_{v-1} \\ g_{v} \end{pmatrix}$$

$$+q^3\begin{pmatrix} 0 & 0 & 0 & \frac{248359}{1207491} & 0 \\ 0 & 0 & 0 & \frac{696541}{4730667} & 0 \\ 0 & 0 & 0 & -\frac{2400291}{10274209} & 0 \\ 0 & 0 & 0 & \frac{232992}{3466613} & 0 \\ 0 & 0 & 0 & \frac{1043650}{3466613} & 0 \end{pmatrix}\begin{pmatrix} m_{v+1} \\ m_{v+2} \\ m_{v+3} \\ m_{v+4} \\ m_{v+5} \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} m_{v-4} \\ m_{v-3} \\ m_{v-2} \\ m_{v-1} \\ m_{v} \end{pmatrix} + q^3\begin{pmatrix} 0 & 0 & 0 & -\frac{15938}{1207491} & 0 \\ 0 & 0 & 0 & \frac{37804}{4730667} & 0 \\ 0 & 0 & 0 & \frac{96918}{10274209} & 0 \\ 0 & 0 & 0 & -\frac{6912}{3466613} & 0 \\ 0 & 0 & 0 & \frac{54740}{3466613} & 0 \end{pmatrix}\begin{pmatrix} w_{d+1} \\ w_{d+2} \\ w_{d+3} \\ w_{d+4} \\ w_{d+5} \end{pmatrix} +$$

$$q^3\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} w_{d-4} \\ w_{d-3} \\ w_{d-2} \\ w_{d-1} \\ w_{d} \end{pmatrix}$$

where

$$D_2^{(3)} = \begin{pmatrix} 1 & -\frac{3474631}{1609988} & \frac{1299241}{1207491} & 0 & 0 \\ \frac{504332}{1576889} & 1 & -\frac{2048940}{1576889} & 0 & 0 \\ \frac{1630755}{10274209} & -\frac{47073879}{41096836} & 1 & 0 & 0 \\ -\frac{88832}{3466613} & \frac{561168}{3466613} & -\frac{3946752}{3466613} & 1 & 0 \\ -\frac{61545}{3466613} & -\frac{221875}{6933226} & -\frac{3409590}{3466613} & 0 & 1 \end{pmatrix}, D_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{396965}{4829964} \\ 0 & 0 & 0 & 0 & -\frac{32281}{1576889} \\ 0 & 0 & 0 & 0 & -\frac{545977}{41096836} \\ 0 & 0 & 0 & 0 & \frac{7803}{3466613} \\ 0 & 0 & 0 & 0 & -\frac{15261}{6933226} \end{pmatrix},$$

$$U_2^{(3)} = \begin{pmatrix} -\frac{6933226}{10867419} & 0 & 0 & \frac{11945267}{21734838} & 0 \\ 0 & \frac{3466613}{3153778} & 0 & \frac{1510807}{3153778} & 0 \\ 0 & 0 & \frac{20799678}{10274209} & -\frac{23766471}{20548418} & 0 \\ 0 & 0 & 0 & \frac{3059700}{3466613} & 0 \\ 0 & 0 & 0 & \frac{6943965}{3466613} & 0 \end{pmatrix}, \ H_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & \frac{3874571}{7244946} & 0 \\ 0 & 0 & 0 & \frac{673941}{1576889} & 0 \\ 0 & 0 & 0 & \frac{17292573}{20548418} & 0 \\ 0 & 0 & 0 & -\frac{1188360}{3466613} & 0 \\ 0 & 0 & 0 & \frac{22545}{3466613} & 0 \end{pmatrix},$$

$$C_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & \frac{248359}{1207491} & 0 \\ 0 & 0 & 0 & \frac{696541}{4730667} & 0 \\ 0 & 0 & 0 & -\frac{2400291}{10274209} & 0 \\ 0 & 0 & 0 & \frac{232992}{3466613} & 0 \\ 0 & 0 & 0 & \frac{1043650}{3466613} & 0 \end{pmatrix} \ and \ B_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{15938}{1207491} & 0 \\ 0 & 0 & 0 & \frac{37804}{4730667} & 0 \\ 0 & 0 & 0 & \frac{96918}{10274209} & 0 \\ 0 & 0 & 0 & -\frac{6912}{3466613} & 0 \\ 0 & 0 & 0 & \frac{54740}{3466613} & 0 \end{pmatrix}$$

The first characteristic polynomial is presented as

$$\begin{aligned} \mu(\varepsilon) &= \det\left(\varepsilon D_2^{(3)} - D_1^{(3)}\right) \\ &= \left|\varepsilon D_2^{(3)} - D_1^{(3)}\right| = 0. \end{aligned} \quad (17)$$

$$\mu(\varepsilon)=\left|\varepsilon\begin{pmatrix}1 & -\frac{3474631}{1609988} & \frac{1299241}{1207491} & 0 & 0\\ \frac{504332}{1576889} & 1 & -\frac{2048940}{1576889} & 0 & 0\\ \frac{1630755}{10274209} & -\frac{47073879}{41096836} & 1 & 0 & 0\\ -\frac{88832}{3466613} & \frac{561168}{3466613} & -\frac{3946752}{3466613} & 1 & 0\\ -\frac{61545}{3466613} & -\frac{221875}{6933226} & -\frac{3409590}{3466613} & 0 & 1\end{pmatrix}-\begin{pmatrix}0 & 0 & 0 & 0 & \frac{396965}{4829964}\\ 0 & 0 & 0 & 0 & -\frac{32281}{1576889}\\ 0 & 0 & 0 & 0 & -\frac{545977}{41096836}\\ 0 & 0 & 0 & 0 & \frac{7803}{3466613}\\ 0 & 0 & 0 & 0 & -\frac{15261}{6933226}\end{pmatrix}\right|=$$

$$\left|\begin{pmatrix}\varepsilon & -\frac{3474631}{1609988}\varepsilon & \frac{1299241}{1207491}\varepsilon & 0 & 0\\ \frac{504332}{1576889}\varepsilon & \varepsilon & -\frac{2048940}{1576889}\varepsilon & 0 & 0\\ \frac{1630755}{10274209}\varepsilon & -\frac{47073879}{41096836}\varepsilon & \varepsilon & 0 & 0\\ -\frac{88832}{3466613}\varepsilon & \frac{561168}{3466613}\varepsilon & -\frac{3946752}{3466613}\varepsilon & \varepsilon & 0\\ -\frac{61545}{3466613}\varepsilon & -\frac{221875}{6933226}\varepsilon & -\frac{3409590}{3466613}\varepsilon & 0 & \varepsilon\end{pmatrix}-\begin{pmatrix}0 & 0 & 0 & 0 & \frac{396965}{4829964}\\ 0 & 0 & 0 & 0 & -\frac{32281}{1576889}\\ 0 & 0 & 0 & 0 & -\frac{545977}{41096836}\\ 0 & 0 & 0 & 0 & \frac{7803}{3466613}\\ 0 & 0 & 0 & 0 & -\frac{15261}{6933226}\end{pmatrix}\right|$$

$$\Rightarrow\mu(\varepsilon)=\begin{pmatrix}\varepsilon & -\frac{3474631}{1609988}\varepsilon & \frac{1299241}{1207491}\varepsilon & 0 & -\frac{396965}{4829964}\\ \frac{504332}{1576889}\varepsilon & \varepsilon & -\frac{2048940}{1576889}\varepsilon & 0 & \frac{32281}{1576889}\\ \frac{1630755}{10274209}\varepsilon & -\frac{47073879}{41096836}\varepsilon & \varepsilon & 0 & \frac{545977}{41096836}\\ -\frac{88832}{3466613}\varepsilon & \frac{561168}{3466613}\varepsilon & -\frac{3946752}{3466613}\varepsilon & \varepsilon & -\frac{7803}{3466613}\\ -\frac{61545}{3466613}\varepsilon & -\frac{221875}{6933226}\varepsilon & -\frac{3409590}{3466613}\varepsilon & 0 & \varepsilon+\frac{15261}{6933226}\end{pmatrix}$$

Using Maple (18) software, we obtained

$$\mu(\varepsilon)=\frac{10094620781085960}{123037159931102549}\varepsilon^5+\frac{10453053312188760}{123037159931102549}\varepsilon^4$$

$$\Rightarrow\frac{10094620781085960}{123037159931102549}\varepsilon^5+\frac{10453053312188760}{123037159931102549}\varepsilon^4=0$$

$$\Rightarrow\varepsilon_1=-\frac{3589703}{3466613},\varepsilon_2=0,\varepsilon_3=0,\varepsilon_4=0,\varepsilon_5=0.$$

Since $|\varepsilon_i|<1, i=1,2,3,4$, thus (14) is zero stable.

3.4. Convergence

Since (10), (12) and (14) are both consistent and zero stable as shown in section 3.3 which satisfied the necessary and sufficient condition for convergence of a numerical method, therefore the proposed method is convergent.

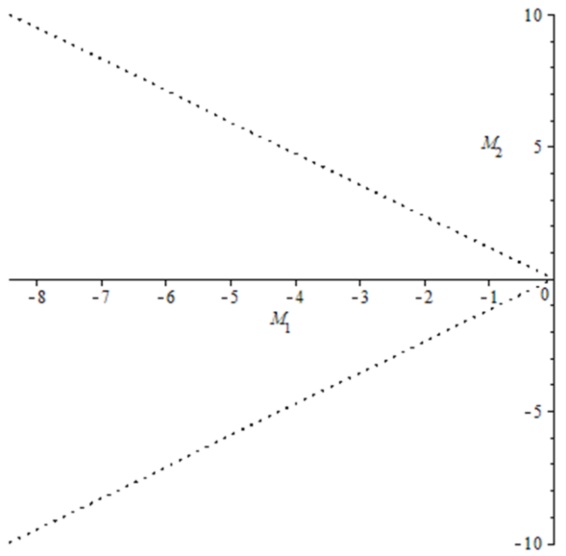

Fig. 1. Region of L- stability (ETDBBDFM) in (10)

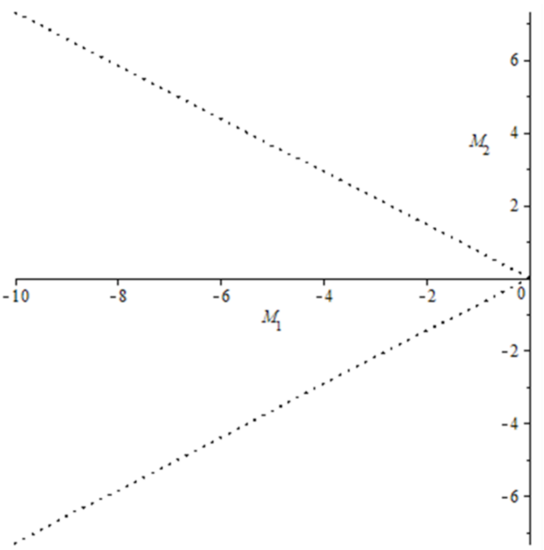

Fig. 2. Region of L- stability (ETDBBDFM) in (12)

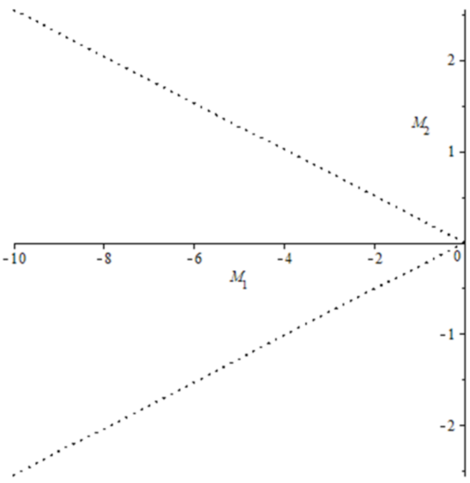

Fig. 3. Region of L- stability (ETDBBDFM) in (14)

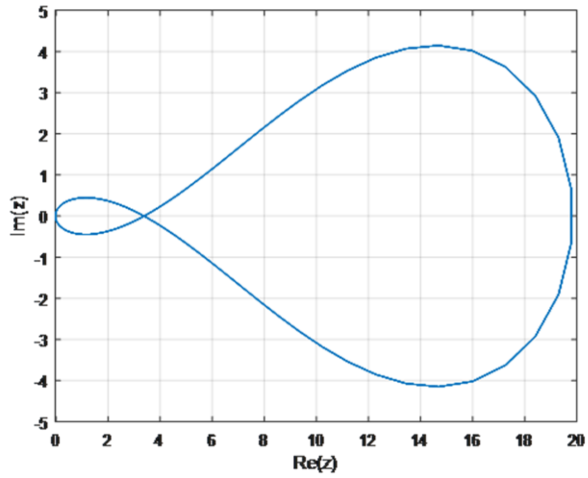

Fig. 4. Region of E- stability (ETDBBDFM) in (10)

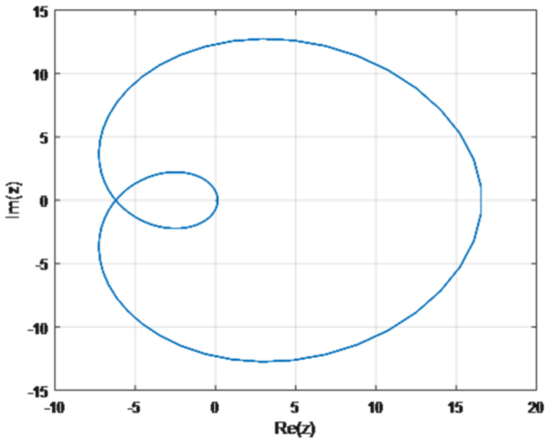

Fig. 5. Region of E- stability (ETDBBDFM) in (12)

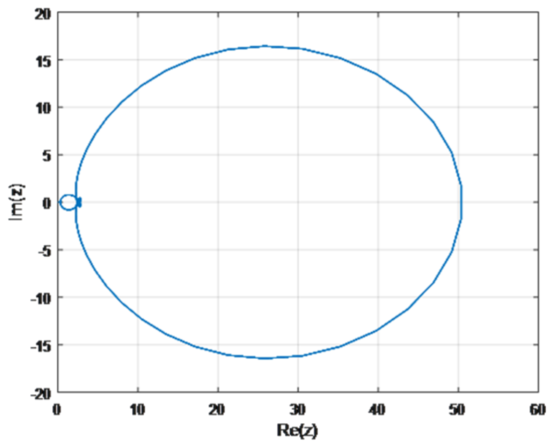

Fig. 6. Region of E-stability (ETDBBDFM) in (14)

Table 1. Comparison between the Maximum Absolute Errors of ETDBBDFM $k = 2, 3$ and 4 and [5, 6] for constant step size h = 0.01 Using Problem 1.

COMPUTATIONAL METHOD	COMPARED MAXEs WITH [5, 6]
ETDBBDFM MAXE for k = 2	0.695891007
ETDBBDFM MAXE for k = 3	0.695947314
ETDBBDFM MAXE for k = 4	0.695956686
TDBBDFM MAXE for k = 2	3.44E-03
TDBBDFM MAXE for k = 3	6.32E-03
TDBBDFM MAXE for k = 4	9.64E-03
ESDBBDFM MAXE for k = 2	5.04E-02
ESDBBDFM MAXE for k = 3	6.69E-02
ESDBBDFM MAXE for k = 4	7.09E-02

CPU time of ETDBBDFM for k = 2 is 0.421s, k = 3 is 0.305s and k = 4 is 0.210s

Table 2. Comparison between the Maximum Absolute Errors of ETDBBDFM $k = 2, 3$ and 4 and [5, 6] for constant step size h = 0.01 Using Problem 2.

COMPUTATIONAL METHOD	COMPARED MAXEs WITH [5, 6]
ETDBBDFM MAXE for k = 2	0.840003646
ETDBBDFM MAXE for k = 3	0.846344125
ETDBBDFM MAXE for k = 4	0.852348729
TDBBDFM MAXE for k = 2	3.44E-03
TDBBDFM MAXE for k = 3	6.29E-03
TDBBDFM MAXE for k = 4	9.64E-03
ESDBBDFM MAXE for k = 2	2.08E-03
ESDBBDFM MAXE for k = 3	1.91E-03
ESDBBDFM MAXE for k = 4	3.43E-03

CPU time of ETDBBDFM for k = 2 is 0.422s, k = 3 is 0.311s and k = 4 is 0.236s

3.5. Region of Absolute Stability

The regions of absolute stability of the numerical methods for DDEs are investigated. We arrived at the L- and E- stability by integrating (10), (12) and (14) to the DDEs of this form

$$y'(t) = ay(t) + by(t-\tau), \quad t \geq t_0, \tau > 0$$

$$y(t) = u(t) \quad t \leq t_0 \tag{18}$$

where $u(t)$is the initial function, a, b are complex coefficients, $\tau = vq, v\epsilon Z^+$, q is the step size and , $q = \frac{\tau}{v}$, v is a positive integer. Let $M_1 = qa$ and $M_2 = qb$, then the L- and E- stability of (10), (12) and (14) are investigated, plotted and presented in figure 1 to 6 below using of Maple 18 and MATLAB.

The L- stability regions in Figs 1 to 3 lie inside the open-ended region while the E- stability regions in Figs 4 to 6 lie inside the enclosed region.

4. Application of the Method

In this section, some first-order delay differential equations shall be solved using (10), (12) and (14) of the discrete schemes been developed. The delay term shall be evaluated by implementing the expression of sequence formed by [4] and the time it took the Central Processing Unit (CPUT) to produce the approximate solutions of each step number of the proposed method shall be taken into consideration.

4.1. Numerical Results

Problem 1

$y'(t) = -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3}), \; 0 \leq t \leq 3$

$y(t) = 1 + e^{-3t}, t \leq 0$

Exact solution: $y(t) = 1 + e^{-3t}$

Problem 2

$y'(t) = -y(t - 1 + e^{-t}) + \sin(t - 1 + e^{-t}) + \cos(t), 0 \leq t \leq 3$

$y(t) = \sin(t), t \leq 0$

Exact Solution: $y(t) = \sin(t)$

The notations used in the table are stated as below:

ETDBBDFM = Extended Third Derivative Block Backward Differentiation Formulae Method for step numbers $k = 2, 3$ *and* 4.

TDBBDFM = Third Derivative Block Backward Differentiation Formulae Method for step numbers $k = 2, 3$ *and* 4.[5]

ESDBBDFM = Extended Second Derivative Block Backward Differentiation Formulae Method for step numbers $k = 2, 3$ *and* 4.[6]

MAXE = Maximum Error.

These two examples were solved using the discrete schemes in (10), (12) and (14), and the maximum error of the results obtained were compared with other existing methods to ascertain it advantage as shown below

5. Conclusion

After proven that the proposed method is convergent, L- and E- stable, the results obtained and analyzed from the application of the proposed method in solving some first order DDE revealed that the ETDBBDFM for $k = 4$ performed better and faster than the step numbers $k = 3$ and $k = 2$ respectively. Also, the comparison of the proposed method with other existing methods showed that ETDBBDFM for $k = 4$ performed better and faster as displayed in tables 1 to 2. Hence, it is recommended that ETDBBDFM for step numbers $k = 2, 3$ *and* 4 are suitable for solving DDEs. Further research should be looked-into for step number $k = 5, 6, 7 \ldots$ on the derivation of discrete schemes of ETDBBDFM for approximate solutions of DDEs without the use of interpolation formula in obtaining the delay argument.

Conflict of interests

The authors declare that they have no conflict of interests.

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