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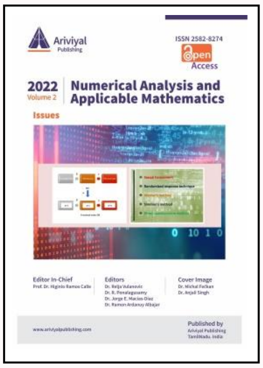

A Numerical Technique for Solving Variable Order Fractional Differential-Integral Equations based on Shifted Fractional Jacobi–Gauss Polynomials

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Abstract: In this manuscript, we display the following coupled differential-integral equations including the Caputo fractional operator of variable-orders: ${}^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p_1{}'(x) + \int_0^x p_1(\upsilon)d\upsilon = \theta_1(x)$, ${}^{Caputo}D_{\varrho_2(x)}[p_2(x)] + p_2{}'(x) + \int_0^x p_2(\upsilon)d\upsilon = \theta_2(x)$, where $\theta_1(x), \theta_2(x)$ are considered the linear and nonlinear functions and $0 < \varrho_i(x) \leq 1, i = 1,2$. To solve numerically these equations by a numerical method based on the shifted Jacobi-Gauss collocation scheme is used. Using this numerical method a system of algebraic equations is constructed. We solve this system with a recursive method in the nonlinear case and we solve it in linear case with algebraic formulas. Finally, for the high performance of the suggested method three Examples are illustrated.

Keywords: Coupled differential-integral equation; Caputo fractional operator; Shifted fractional Jacobi collocation method; Variable-order

1. Introduction

A coupled differential-integral equation including the Caputo fractional operator is introduced as follows:

$$ {}^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p_1{}'(x) + \int_0^x p_1(\upsilon)d\upsilon = \theta_1(x) \tag{1}$$

$$ {}^{Caputo}D_{\varrho_2(x)}[p_2(x)] + p_2{}'(x) + \int_0^x p_2(\upsilon)d\upsilon = \theta_2(x), \tag{2}$$

the initial condition for this types of equations (1) and (2) are given by:

$$p_2(0) = p_2'(0) = 0, p_1(0) = p_1'(0) = 0$$

where ${}^{Caputo}D_{\varrho_i(x)}, i = 1,2$ are the Caputo fractional operators of variable orders $0 < \varrho_i(x) \leq 1, i = 1,2$ which are given by:

$$ {}^{Caputo}D_{\varrho_i(x)}p(x) =_{i=1,2} \begin{cases} p'(x), & \varrho_i(x) = 1, \\ \frac{1}{\Gamma(1-\varrho_i(x))}\int_0^x (x-\upsilon)^{-\varrho_i(x)}p'(\upsilon), & 0 < \varrho_i(x) \leq 1. \end{cases} \tag{3}$$

In the coupled differential-integral equations (1)(2), $p_1(x), p_2(x)$ and $\theta_1(x), \theta_2(x)$ are indistincted and definited respectively. The main reason for choosing these kind of differential-integral equations is because of their role and applications in various fields as computational and mathematical sciences,[14,2,16] engineering sciences which are modeled using these types of equations,[12,23,24] physics[6,11,17] and mathematical models in chemistry.[18,26]

The main purpose of this paper is to obtain the solution of the coupled differential-integral equations are introduced in (1),(2) by using a numerical method that this numerical method is called a shifted Jacobi-Gauss collocation algorithm. This numerical method contains a shifted Jacobi polynomials. Obtaining a solution by using different numerical methods was studied by some authors in the mathematical field., that in this paper a few numerical methods are mentioned, for Example, in[13] was studied a numerical method based on Chebyshev cardinal wavelets, the new algorithm base on the discretization method to obtain solution of the fractional functional integral equations of variable-order in[15] was discussed, the explicit and implicit Euler methods,[28] the Legendre wavelets method,[4] the numerical method base on Laplace and Sumudu transform methods,[8] the Bezier curve method,[9] the Adams-Bashforth-Moulton scheme,[5] the optimization method[25] and other ways.[19,20,10,21,3]

So, this article is divided into four sections as follows. In Section 2, Lemma and definitions of Jacobi polynomials, shifted Jacobi polynomials and their properties are introduced. In Section 3, approximation function and the numerical method to obtaining solution of the coupled differential-integral equation are described. In order to accuracy of the presented method four examples in Section 4 are described.

2. Some main lemma and definitions about the Jacobi polynomials and shifted Jacobi polynomials

This section deals with some definitions about the Jacobi polynomials and shifted Jacobi polynomials and Lemma which will be applied in next section. Let $\mu,\nu > -1, x \in [-1,1]$. Then the Jacobi polynomial of degree n is defined by [7]:

$$\mathfrak{P}_{n+1}^{\mu,\nu}(x) = (a_n^{\mu,\nu}x - b_n^{\mu,\nu})\mathfrak{P}_n^{\mu,\nu}(x) - c_n^{\mu,\nu}\mathfrak{P}_{n-1}^{\mu,\nu}(x), \tag{4}$$

where for $n = 0,1$ the functions $\mathfrak{P}_0^{\mu,\nu}(x)$ and $\mathfrak{P}_1^{\mu,\nu}(x)$ are defined by:

$$\mathfrak{P}_0^{\mu,\nu}(x) = 1, \mathfrak{P}_1^{\mu,\nu}(x) = \frac{(\mu+\nu+2)x+\mu-\nu}{2}, \tag{5}$$

and the coefficients $a_n^{\mu,\nu}, b_n^{\mu,\nu}, c_n^{\mu,\nu}$ are given by:

$$a_n^{\mu,\nu} = \frac{(2n+\mu+\nu+1)(2n+\mu+\nu+2)}{(2n+2)(n+\mu+\nu+1)}, b_n^{\mu,\nu} = \frac{(\nu^2-\mu^2)(2n+\mu+\nu+1)}{(2n+2)(n+\mu+\nu+1)(2n+\mu+\nu)}, c_n^{\mu,\nu} = \frac{(n+\mu)(n+\nu)(2n+\mu+\nu+2)}{(n+1)(n+\mu+\nu+1)(2n+\mu+\nu)}. \tag{6}$$

Also, the Jacobi polynomial has a finite series as follows:

$$\mathfrak{P}_n^{\mu,\nu}(x) = \frac{\Gamma(\mu+n+1)}{n!\Gamma(\mu+\nu+n+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\mu+\nu+k+n+1)}{\Gamma(\mu+k+1)}\left(\frac{x-1}{2}\right)^k, \tag{7}$$

where $\Gamma(.)$ is the gamma function. Let $\mu,\nu > -1$. Then the shifted Jacobi polynomial of degree n is defined by[7]:

$$\mathbb{P}_n^{\mu,\nu}(x) = \mathfrak{P}_n^{\mu,\nu}(\underbrace{2x-1}_{2x-1\in[-1,1]}) = \frac{\Gamma(\mu+n+1)}{n!\Gamma(\mu+\nu+n+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\mu+\nu+k+n+1)}{\Gamma(\mu+k+1)}(x-1)^k = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)}\sum_{k=0}^{n}(-1)^{n-k}\frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}x^k. \tag{8}$$

Also, the shifted Jacobi polynomial respect to the weight function $\omega^{\mu,\nu}(x) = (1-x)^\mu x^\nu$ is orthogonal, that the orthogonality condition is given as:

$$\langle\mathbb{P}_n^{\mu,\nu}(x), \mathbb{P}_m^{\mu,\nu}(x)\rangle_{\omega^{\mu,\nu}(x)} = \int_0^1 \omega^{\mu,\nu}(x)\mathbb{P}_n^{\mu,\nu}(x)\mathbb{P}_m^{\mu,\nu}(x)dx = \begin{cases}0, & n \neq m,\\ \frac{\Gamma(n+\mu+\nu+1)}{n!\Gamma(2n+\mu+\nu+1)}, & n = m,\end{cases} \tag{9}$$

where $\langle .,. \rangle$ is the inner product. Let $0 < \varrho_i \leq 1, i = 1,2$. Then for any $\zeta \geq -1$ the Caputo fractional operator of variable order $\varrho_i, i = 1,2$ is given by[27]:

$$^{Caputo}D_{\varrho_i(x)}x^\zeta = \begin{cases}0, & \zeta = \text{constant},\\ \frac{\Gamma(\zeta+1)}{\Gamma(\zeta-\varrho_i(x)+1)}x^{\zeta-\varrho_i(x)}, & \zeta \geq -1,\\ \frac{d^m}{dx^m}(x^\zeta), & \varrho_i(x) = m, m \in \mathbb{N}.\end{cases} \tag{10}$$

So, applying the relation (10) on (8), for $i = 1,2$ we obtain:

$$^{Caputo}D_{\varrho_i(x)}\mathbb{P}_n^{\mu,\nu}(x) = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)}\sum_{k=0}^{n}(-1)^{n-k}\frac{\Gamma(k+1)}{\Gamma(k-\varrho_i(x)+1)}\times\frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}x^{k-\varrho_i(x)} = \sum_{k=0}^{n}\Omega_{k,n}^{\mu,\nu}x^{k-\varrho_i(x)}, \tag{11}$$

$$\underbrace{^{Caputo}D_{\varrho_i(x)}\mathbb{P}_n^{\mu,\nu}(x)}_{\text{For } \varrho_i(x)=1, m=1} = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)}\sum_{k=1}^{n}(-1)^{n-k}\frac{k\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}x^{k-1} = \sum_{k=1}^{n}\Psi_{k,n}^{\mu,\nu}x^{k-1} \tag{12}$$

where $\Omega_{k,n}^{\mu,\nu} = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)}\times(-1)^{n-k}\frac{\Gamma(k+1)}{\Gamma(k-\varrho_i(x)+1)}\times\frac{\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}$ and $\Psi_{k,n}^{\mu,\nu} = \frac{\Gamma(\nu+n+1)}{\Gamma(\mu+\nu+n+1)}\times(-1)^{n-k}\frac{k\Gamma(\mu+\nu+k+n+1)}{k!(n-k)!\Gamma(\nu+k+1)}$.

3. The function approximation and proposed method algorithm

Due to the orthogonality condition is presented in (9) about the shifted Jacobi polynomial, any $p_1(x), p_2(x) \in L^2[0,1]$ can be approximated in terms of the shifted Jacobi polynomial as follows:

$$p_1(x) = \sum_{k=0}^{\infty} a_k \mathbb{P}_k^{\mu,\nu}(x), \tag{13}$$
$$p_2(x) = \sum_{k=0}^{\infty} b_k \mathbb{P}_k^{\mu,\nu}(x), \tag{14}$$

considering the first n sentence of the relations (13), (14), we have:

$$p_1(x) \simeq p_{1,n}(x) = \sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(x), \tag{15}$$
$$p_2(x) \simeq p_{2,n}(x) = \sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(x), \tag{16}$$

where the coefficients a_j, b_j are calculated as:

$$a_j = \langle \mathbb{P}_n^{\mu,\nu}(x), p_1(x) \rangle_{\omega^{\mu,\nu}(x)} = \frac{\int_0^1 \omega^{\mu,\nu}(x)\mathbb{P}_n^{\mu,\nu}(x) p_1(x)dx}{\|\mathbb{P}_n^{\mu,\nu}(x)\|^2}, \tag{17}$$
$$b_j = \langle \mathbb{P}_n^{\mu,\nu}(x), p_2(x) \rangle_{\omega^{\mu,\nu}(x)} = \frac{\int_0^1 \omega^{\mu,\nu}(x)\mathbb{P}_n^{\mu,\nu}(x) p_2(x)dx}{\|\mathbb{P}_n^{\mu,\nu}(x)\|^2}. \tag{18}$$

Theorem 1. Let $p_{1,n}$ be as an approximation of the function $p_1(x)$, that it satisfy in the equation(1) and the following conditions for $\lambda_1, \lambda_2 \in (0,1)$ hold:

$$\| p_{1,n}(x) - p_{1,n-1}(x) \| \leq \lambda_1 \| p_{1,n-1}(x) - p_{1,n-2}(x) \| \tag{19}$$
$$\| p'_{1,n}(x) - p'_{1,n-1}(x) \| \leq \lambda_2 \| p'_{1,n-1}(x) - p'_{1,n-2}(x) \|. \tag{20}$$

Then the sequence $p_{1,n}$ to $p_1(x)$ converges, that $p_1(x)$ is the exact solution of the equation(1),
Proof. Suppose $\mathbf{R}_n(x)$ be a residual function, which is defined as:

$$\mathbf{R}_n(x) = {}^{Caputo} D_{\varrho_1(x)}[p_{1,n-1}(x)] + p_{1,n-1}{}'(x) + \int_0^x p_{1,n-1}(v)dv - \theta_1(x),$$
$$\mathbf{R}_{n-1}(x) = {}^{Caputo} D_{\varrho_1(x)}[p_{1,n-2}(x)] + p_{1,n-2}{}'(x) + \int_0^x p_{1,n-2}(v)dv - \theta_1(x). \tag{21}$$

Then, we have:

$$\| \mathbf{R}_n(x) - \mathbf{R}_{n-1}(x) \| = \| {}^{Caputo} D_{\varrho_1(x)}[p_{1,n-1}(x) - p_{1,n-2}(x)] + (p_{1,n-1}{}'(x) - p_{1,n-2}{}'(x)) + \int_0^x (p_{1,n-1}(v) - p_{1,n-2}(v)dv) \|$$

$$= \| \frac{1}{\Gamma(1-\varrho_1(x))} \int_0^x (x-v)^{-\varrho_1(x)} [p'_{1,n-1}(v) - p'_{1,n-2}(v)]dv + (p_{1,n-1}{}'(x) - p_{1,n-2}{}'(x)) + \int_0^x (p_{1,n-1}(v) - p_{1,n-2}(v)dv) \|$$

$$\leq \frac{1}{\Gamma(1-\varrho_1(x))} \int_0^x (x-v)^{-\varrho_1(x)} \| p'_{1,n-1}(v) - p'_{1,n-2}(v) \| dv + \| p_{1,n-1}{}'(x) - p_{1,n-2}{}'(x) \| + \int_0^x \| p_{1,n-1}(v) - p_{1,n-2}(v) \| dv$$
$$= \frac{x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \int_0^x \| p'_{1,n-1}(v) - p'_{1,n-2}(v) \| dv + \| p_{1,n-1}{}'(x) - p_{1,n-2}{}'(x) \| + \int_0^x \| p_{1,n-1}(v) - p_{1,n-2}(v) \| dv. \tag{22}$$

Using (19), (20), we get:

$$\| \mathbf{R}_n(x) - \mathbf{R}_{n-1}(x) \| \leq \frac{\lambda_2 x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \int_0^x \| p'_{1,n-2}(v) - p'_{1,n-3}(v) \| dv + \lambda_2 \| p_{1,n-2}{}'(x) - p_{1,n-3}{}'(x) \| + \lambda_1 \int_0^x \| p_{1,n-2}(v) - p_{1,n-3}(v) \| dv$$
$$\vdots$$
$$\leq \frac{\lambda_2^{n-2} x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \int_0^x \| p'_{1,1}(v) - p'_{1,0}(v) \| dv + \lambda_2^{n-2} \| p_{1,1}{}'(x) - p_{1,0}{}'(x) \| + \lambda_1^{n-2} \int_0^x \| p_{1,1}(v) - p_{1,0}(v) \| dv$$
$$\leq \frac{\lambda_2^{n-2} x^{2-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \max_{x\in[0,1]} \| p'_{1,1}(x) - p'_{1,0}(x) \| + \lambda_2^{n-2} \max_{x\in[0,1]} \| p_{1,1}{}'(x) - p_{1,0}{}'(x) \| + \lambda_1^{n-2} \max_{x\in[0,1]} \| p_{1,1}(x) - p_{1,0}(x) \|. \tag{23}$$

So, $\| \mathbf{R}_n(x) - \mathbf{R}_{n-1}(x) \| \to 0$ when $n \to \infty$, then $\mathbf{R}_n(x)$ a Cauchy sequence in $L^2[0,1]$ and $L^2[0,1]$ is complete, therefore the sequence $\mathbf{R}_n(x)$ is convergent that is

$$\lim_{n\to\infty} \mathbf{R}_n(x) = \mathbf{R}(x), \tag{24}$$

where

$$\begin{aligned} \mathbf{R}_n(x) &= {}^{Caputo} D_{\varrho_1(x)}[p_{1,n-1}(x)] + p_{1,n-1}{}'(x) + \int_0^x p_{1,n-1}(v)dv - \theta_1(x), \\ \mathbf{R}(x) &= {}^{Caputo} D_{\varrho_1(x)}[p_1(x)] + p'_1(x) + \int_0^x p_1(v)dv - \theta_1(x). \end{aligned} \tag{25}$$

The proof is complete.

Like the Theorem 1 can be considered a similar process for the equation (2).

Theorem 2.([22]) Suppose $f(x)$ be belongs to the space $\mathbf{H}^\sigma(0,1), \sigma \geq 0$ that $\mathbf{H}^\sigma(0,1)$ is defined by:

$$\mathbf{H}^\sigma(0,1) = \{f \in L^2(0,1): \| f \|_{\mathbf{H}^\sigma(0,1)} = (\textstyle\sum_{k=0}^{\sigma} \| f^{(k)} \|^2_{L^2(0,1)})^{\frac{1}{2}} < \infty\}, \tag{26}$$

and $P_{k,M}$ as the best approximation of f is considered. Then the following inequality is hold:

$$\| f - P_{k,M} \|_{L^2(0,1)} \leq cM^{-\sigma} 2^{-k\sigma} \| f^{(\sigma)} \|^2_{L^2(0,1)}, \tag{27}$$

$$\| f - P_{k,M} \|_{\mathbf{H}^q(0,1)} \leq cM^{2q-\frac{1}{2}-\sigma} 2^{k(q-\sigma)} \| f^{(\sigma)} \|^2_{L^2(0,1)}, q \geq 1. \tag{28}$$

Theorem 3. Suppose $p_1(x)$ is belongs to the space $\mathbf{H}^\sigma(0,1), \sigma \geq 0$ and p_N is the best approximation of $p_1(x)$. Then we have:

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}, \tag{29}$$

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}, q \geq 1, \tag{30}$$

here $\mathbb{E} = p_1(x) - p_N(x)$ is considered.

Proof. Using (1), we obtain

$$\begin{aligned} \| \mathbb{E} \|_{L^2(0,1)} &= \| {}^{Caputo} D_{\varrho_1(x)}[p_1(x) - p_N(x)] + (p_1{}'(x) - p_N{}'(x)) + \int_0^x (p_1(v) - p_N(v)dv) \|_{L^2(0,1)} \\ &= \| \frac{1}{\Gamma(1-\varrho_1(x))} \int_0^x (x-v)^{-\varrho_1(x)}[p'_1(v) - p'_N(v)]dv + (p_1{}'(x) - p_N{}'(x)) + \int_0^x (p_1(v) - p_N(v)dv) \|_{L^2(0,1)} \\ &\leq \frac{1}{\Gamma(1-\varrho_1(x))} \| \int_0^x (x-v)^{-\varrho_1(x)}[p'_1(v) - p'_N(v)]dv \|_{L^2(0,1)} + \| p_1{}'(x) - p_N{}'(x) \|_{L^2(0,1)} \\ &+ \| \int_0^x (p_1(v) - p_N(v)dv) \|_{L^2(0,1)}. \end{aligned} \tag{31}$$

we use $\| f * g \|_{L^2(0,1)} \leq \| f \|_1 \| g \|_{L^2(0,1)}$ for Eq.(??), we have:

$$\begin{aligned} \| \mathbb{E} \|_{L^2(0,1)} &\leq \frac{1}{\Gamma(1-\varrho_1(x))} \| \int_0^x (x-v)^{-\varrho_1(x)}[p'_1(v) - p'_N(v)]dv \|_{L^2(0,1)} + \| p_1{}'(x) - p_N{}'(x) \|_{L^2(0,1)} \\ &+ \| \int_0^x (p_1(v) - p_N(v)dv) \|_{L^2(0,1)} \\ &\leq \frac{x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \| p'_1(x) - p'_N(x) \|_{L^2(0,1)} + \| p_1{}'(x) - p_N{}'(x) \|_{L^2(0,1)} + \| \int_0^x (p_1(v) - p_N(v)dv) \|_{L^2(0,1)} \\ &\leq \frac{x^{1-\varrho_1(x)}}{\Gamma(2-\varrho_1(x))} \| p_1(x) - p_N(x) \|_{L^2(0,1)} + \| p_1(x) - p_N(x) \|_{L^2(0,1)} + x \| p_1(x) - p_N(x) \|_{L^2(0,1)} \\ &\leq \frac{\| p_1(x) - p_N(x) \|_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2 \| p_1(x) - p_N(x) \|_{L^2(0,1)}, \end{aligned} \tag{32}$$

so, we employ Eqs.(27),(28) on the relation (32), we get:

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}, \tag{33}$$

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}}{\Gamma(2-\varrho_1(x))} + 2cN^{2q-\frac{1}{2}-\sigma} \| p_1^{(\sigma)} \|^2_{L^2(0,1)}, q \geq 1. \tag{34}$$

The desired result is achieved.

Theorem 4. Suppose $p_2(x)$ is belongs to the space $\mathbf{H}^{\sigma}(0,1), \sigma \geq 0$ and p_N, is the best approximation of $p_1(x)$. Then we have:

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN\prime^{-\sigma} \| p_1^{(\sigma)} \|_{L^2(0,1)}^2}{\Gamma(2-\varrho_2(x))} + 2cN'^{-\sigma} \| p_2^{(\sigma)} \|_{L^2(0,1)}^2, \tag{35}$$

$$\| \mathbb{E} \|_{L^2(0,1)} \leq \frac{cN\prime^{2q-\frac{1}{2}-\sigma} \| p_2^{(\sigma)} \|_{L^2(0,1)}^2}{\Gamma(2-\varrho_2(x))} + 2cN'^{2q-\frac{1}{2}-\sigma} \| p_2^{(\sigma)} \|_{L^2(0,1)}^2, q \geq 1. \tag{36}$$

Proof. The proof of this Theorem is similar to the proof of the Theorem 3.

3.1. The shifted Jacobi-Gauss collocation algorithm

This subsection describes the proposed method for solving Eqs.(1), (2). By substituting the relations (15) and (16) in Eqs.(1), (2), we obtain

$$\underbrace{{}^{Caputo}D_{\varrho_1(x)}\left[\sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(x)\right] + \frac{d}{dx}\left(\sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(x)\right) + \int_0^x \sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(v)dv = \theta_1(x),}_{the\ initial\ conditon\ for\ this\ types\ of\ euation\ is\ \sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^{n} a_k \mathbb{P}'^{\mu,\nu}_k(0) = 0}$$

$$\sum_{k=0}^{n} a_k {}^{Caputo}D_{\varrho_1(x)} \mathbb{P}_k^{\mu,\nu}(x) + \sum_{k=0}^{n} a_k \mathbb{P}'^{\mu,\nu}_k(x) + \sum_{k=0}^{n} a_k \int_0^x \mathbb{P}_k^{\mu,\nu}(v)dv = \theta_1(x), \tag{37}$$

$$\underbrace{{}^{Caputo}D_{\varrho_2(x)}\left[\sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(x)\right] + \sum_{k=0}^{n} b_k \mathbb{P}'^{\mu,\nu}_k(x) + \int_0^x \sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(v)dv = \theta_2(x),}_{the\ initial\ conditon\ for\ this\ types\ of\ euation\ is\ \sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^{n} b_k \mathbb{P}'^{\mu,\nu}_k(0) = 0}$$

$$\sum_{k=0}^{n} b_k {}^{Caputo}D_{\varrho_1(x)} \mathbb{P}_k^{\mu,\nu}(x) + \sum_{k=0}^{n} b_k \mathbb{P}'^{\mu,\nu}_k(x) + \sum_{k=0}^{n} b_k \int_0^x \mathbb{P}_k^{\mu,\nu}(v)dv = \theta_2(x), \tag{38}$$

by applying Eqs.(11),(12) on the equations (37) and (38), we have:

$$\sum_{k=0}^{n} \sum_{m=0}^{k} a_k \Omega_{m,k}^{\mu,\nu} x^{m-\varrho_1(x)} + \sum_{k=0}^{n} \sum_{m=1}^{k+1} a_k \Psi_{m,k}^{\mu,\nu} x^{m-1} + \sum_{k=0}^{n} \sum_{m=0}^{k} a_k \Upsilon_{m,k}^{\mu,\nu} x^{m+1} = \theta_1(x), \tag{39}$$

$$\sum_{k=0}^{n} \sum_{m=0}^{k} b_k \Omega_{m,k}^{\mu,\nu} x^{m-\varrho_2(x)} + \sum_{k=0}^{n} \sum_{m=1}^{k+1} b_k \Psi_{m,k}^{\mu,\nu} x^{m-1} + \sum_{k=0}^{n} \sum_{m=0}^{k} b_k \Upsilon_{m,k}^{\mu,\nu} x^{m+1} = \theta_2(x), \tag{40}$$

where, $\Upsilon_{m,k}^{\mu,\nu} = (-1)^{k-m} \frac{\Gamma(\nu+k+1)\Gamma(\mu+\nu+m+k+1)}{(m+1)!(k-m)!\Gamma(\nu+m+1)}$.

Now we calculate the equations (39) and (40) in points $x_{\mu,\nu,i} = \frac{x_{\mu,\nu,i}+1}{2}$ which is the nodes of the standard Jacobi-Gauss interpolation in the interval $[-1,1]$ and their introduced in [7]. Then for $i = 1,2, \dots, n$ we get:

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{m=0}^{k} a_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_1(x)} + \sum_{k=0}^{n} \sum_{m=1}^{k+1} a_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^{n} \sum_{m=0}^{k} a_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_1(x_{\mu,\nu,i}), \\ &\sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^{n} a_k \mathbb{P}'^{\mu,\nu}_k(0) = 0. \end{aligned} \tag{41}$$

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{m=0}^{k} b_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_2(x)} + \sum_{k=0}^{n} \sum_{m=1}^{k+1} b_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^{n} \sum_{m=0}^{k} b_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_2(x_{\mu,\nu,i}), \\ &\sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^{n} b_k \mathbb{P}'^{\mu,\nu}_k(0) = 0. \end{aligned} \tag{42}$$

By solving these equations together with the initial conditions, the uncertain coefficients a_k, b_k are obtained. When the functions $\theta_1(x), \theta_2(x)$ are nonlinear, in this case the equations (41) and (42) changes to the following equations:

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{m=0}^{k} a_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_1(x)} + \sum_{k=0}^{n} \sum_{m=1}^{k+1} a_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^{n} \sum_{m=0}^{k} a_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_1(x_{\mu,\nu,i}, p_1(x_{\mu,\nu,i}), p'_1(x_{\mu,\nu,i})), \\ &\sum_{k=0}^{n} a_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^{n} a_k \mathbb{P}'^{\mu,\nu}_k(0) = 0. \end{aligned} \tag{43}$$

$$\begin{aligned} &\sum_{k=0}^{n} \sum_{m=0}^{k} b_k \Omega_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-\varrho_2(x)} + \sum_{k=0}^{n} \sum_{m=1}^{k+1} b_k \Psi_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m-1} + \sum_{k=0}^{n} \sum_{m=0}^{k} b_k \Upsilon_{m,k}^{\mu,\nu} x_{\mu,\nu,i}^{m+1} = \theta_2(x_{\mu,\nu,i}, p_2(x_{\mu,\nu,i}), p'_2(x_{\mu,\nu,i})), \\ &\sum_{k=0}^{n} b_k \mathbb{P}_k^{\mu,\nu}(0) = \sum_{k=0}^{n} b_k \mathbb{P}'^{\mu,\nu}_k(0) = 0. \end{aligned} \tag{44}$$

By applying a recursive method on the relations (43) and (44) can be obtained the uncertain coefficients a_k, b_k.

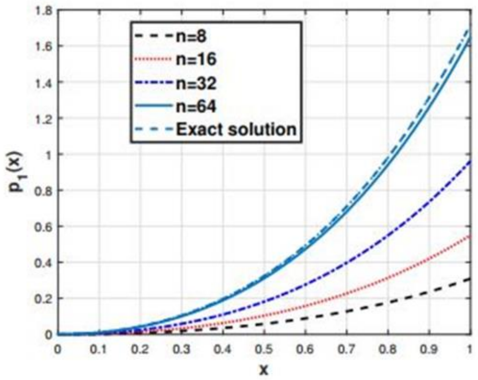

Fig. 1. Numerical and exact solutions of the Example 1 with $\varrho_1(x) = 1 - 0.001x$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$

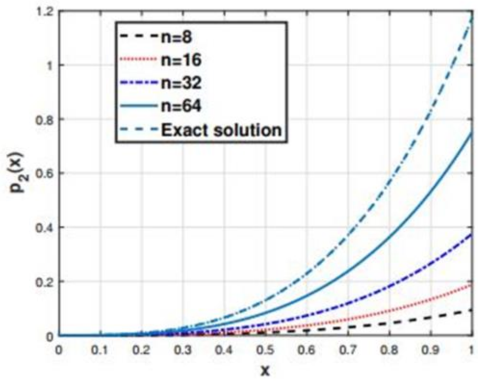

Fig. 2. Numerical and exact solutions of the Example 1 with $\varrho_2(x) = 1 - \sin^2(x)$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$

Table 1. The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_1(x) = 1 - 0.001x$ for Example 1

-34-5 x	$\lvert p_1(x) - p_{1,n}(x)\rvert$		$\lvert p_1(x) - p_{1,n}(x)\rvert$	
	$n = 8$	$n = 16$	$n = 32$	$n = 64$
0.1	$0.96485e - 15$	$0.99077e - 16$	$1.01719e - 17$	$1.04410e - 18$
0.2	$0.86589e - 15$	$0.88993e - 16$	$0.91443e - 17$	$0.93940e - 18$
0.3	$0.77425e - 15$	$0.79650e - 16$	$0.81918e - 17$	$0.84231e - 18$
0.4	$0.68954e - 15$	$0.71009e - 16$	$0.73105e - 17$	$0.75244e - 18$
0.5	$0.61139e - 15$	$0.63033e - 16$	$0.64967e - 17$	$0.66940e - 18$
0.6	$0.53947e - 15$	$0.55688e - 16$	$0.57467e - 17$	$0.59284e - 18$
0.7	$0.47343e - 15$	$0.48940e - 16$	$0.50573e - 17$	$0.52242e - 18$
0.8	$0.41297e - 15$	$0.42757e - 16$	$0.44252e - 17$	$0.45780e - 18$
0.9	$0.35778e - 15$	$0.37110e - 16$	$0.38473e - 17$	$0.39869e - 18$

Table 2. The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_2(x) = 1 - \sin^2(x)$ for Example 1

-34-5 x	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
	n=8	n=16	n=32	n=64
0.1	$1.02256e - 15$	$1.12266e - 17$	$1.22921e - 19$	$1.34242e - 21$
0.2	$1.46252e - 17$	$1.58973e - 19$	$1.72428e - 21$	$1.86639e - 23$
0.3	$0.68232e - 17$	$0.75878e - 19$	$0.84084e - 21$	$0.92869e - 23$
0.4	$0.42820e - 17$	$0.48438e - 19$	$0.54532e - 21$	$0.61123e - 23$
0.5	$0.24707e - 17$	$0.28622e - 19$	$0.32932e - 21$	$0.37658e - 23$
0.6	$0.12620e - 17$	$0.15145e - 19$	$0.17989e - 21$	$0.21170e - 23$
0.7	$0.05314e - 17$	$0.06759e - 19$	$0.08446e - 21$	$0.10393e - 23$
0.8	$0.01572e - 17$	$0.02239e - 19$	$0.03073e - 21$	$0.04091e - 23$
0.9	$0.00196e - 17$	$0.00383e - 19$	$0.00663e - 21$	$0.01053e - 23$

4. Numerical Examples

This section examines three examples by using the proposed method to show its performance and accuracy.

4.1. Example 1

We consider the following coupled differential-integral equation of order $\varrho_1(x) = 1 - 0.001x, \varrho_2(x) = 1 - \sin^2(x)$:

$$
\begin{aligned}
&{}^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p_1{}'(x) + \int_0^x p_2(\upsilon)d\upsilon = \theta_1(x),\\
&p_1(0) = p_1{}'(0) = 0, \qquad (45)\\
&{}^{Caputo}D_{\varrho_2(x)}[p_2(x)] + p_2{}'(x) + \int_0^x p_1(\upsilon)d\upsilon = \theta_2(x),\\
&p_2(0) = p_2{}'(0) = 0, \qquad (46)
\end{aligned}
$$

where $\theta_1(x) = x^2\cosh(x) - 2t\sinh(x) + 2\cosh(x) + 2(e^x + 2xe^x - 1)$ and $\theta_2(x) = 4x\cosh(x) + 2x^2\cosh(x) + e^x(x - 1)$. The analytical solutions for these questions are $p_1(x) = xe^x - x, p_2(x) = x^2\sinh(x)$. This Example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figs. 1 and 2. In the Tables 1 and 2, the absolute error between the approximate and analytical solutions are displayed.

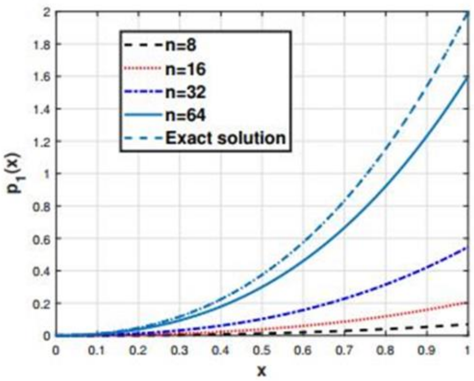

Fig. 3. Numerical and exact solutions of the Example 2 with $\varrho_1(x) = \sin(\frac{x}{3})$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$

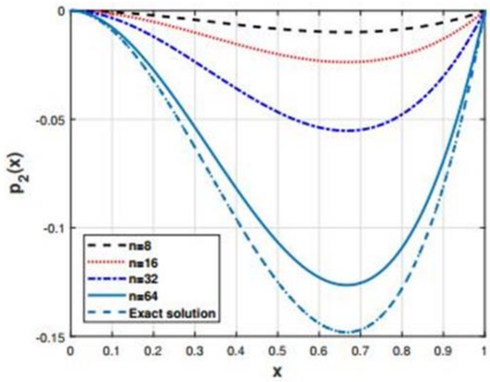

Fig. 4. Numerical and exact solutions of the Example 2 with $\varrho_2(x) = \frac{x}{3}$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$

Table 3. The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_1(x) = \sin(\frac{x}{3})$ for Example 2

-34-5 x	$\lvert p_1(x) - p_{1,n}(x)\rvert$		$\lvert p_1(x) - p_{1,n}(x)\rvert$	
	n=8	n=16	n=32	n=64
0.1	$0.20113e-15$	$0.20639e-16$	$0.21173e-17$	$0.21717e-18$
0.2	$0.18095e-15$	$0.18586e-16$	$0.19087e-17$	$0.19595e-18$
0.3	$0.16210e-15$	$0.16669e-16$	$0.17136e-17$	$0.17611e-18$
0.4	$0.14456e-15$	$0.14883e-16$	$0.15317e-17$	$0.15760e-18$
0.5	$0.12827e-15$	$0.13223e-16$	$0.13626e-17$	$0.14037e-18$
0.6	$0.11319e-15$	$0.11685e-16$	$0.12058e-17$	$0.12439e-18$
0.7	$0.09928e-15$	$0.10265e-16$	$0.10609e-17$	$0.10960e-18$
0.8	$0.08649e-15$	$0.08959e-16$	$0.09275e-17$	$0.09598e-18$
0.9	$0.07479e-15$	$0.07762e-16$	$0.08051e-17$	$0.08347e-18$

Table 4. The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_2(x) = \frac{x}{3}$ for Example 2

-34-5 x	$\lvert p_2(x) - p_{2,n}(x)\rvert$		$\lvert p_2(x) - p_{2,n}(x)\rvert$	
	n=8	n=16	n=32	n=64
0.1	$2.18198e-15$	$2.26887e-16$	$2.35518e-17$	$2.44075e-18$
0.2	$1.83178e-15$	$1.91941e-16$	$2.00709e-17$	$2.09466e-18$
0.3	$1.48495e-15$	$1.57079e-16$	$1.65732e-17$	$1.74436e-18$
0.4	$1.15160e-15$	$1.23312e-16$	$1.31596e-17$	$1.39996e-18$
0.5	$0.84181e-15$	$0.91649e-16$	$0.99312e-17$	$1.07154e-18$
0.6	$0.56569e-15$	$0.63101e-16$	$0.69891e-17$	$0.76923e-18$
0.7	$0.33335e-15$	$0.38678e-16$	$0.44342e-17$	$0.50311e-18$
0.8	$0.15489e-15$	$0.19390e-16$	$0.23676e-17$	$0.28329e-18$
0.9	$0.04040e-15$	$0.06247e-16$	$0.08902e-17$	$0.11987e-18$

4.2. Example 2

Consider the following coupled differential-integral equation with $\varrho_1(x) = \sin(\frac{x}{3}), \varrho_2(x) = \frac{x}{3}$:

$$^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p_1'(x) + \int_0^x p_1(v)dv = 4x(x+1) + 2x^2 + \frac{x^4}{4} + \frac{x^3}{3},$$
$$p_1(0) = p_1'(0) = 0, \tag{47}$$

$$^{Caputo}D_{\varrho_2(x)}[p_2(x)] + p_2'(x) + \int_0^x p_2(v)dv = 4x(x-1) + 2x^x + \frac{x^4}{4} - \frac{x^3}{3},$$
$$p_2(0) = p_2'(0) = 0, \tag{48}$$

the exact solutions of Eqs.(47),(48) are $p_1(x) = x^2(x+1), p_2(x) = x^2(x-1)$. This Example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figs. 3 and 4. In the Tables 3 and 4, the absolute error between the approximate and analytical solutions are displayed.

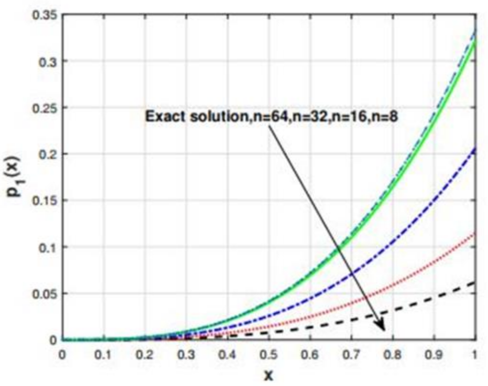

Fig. 5. Numerical and exact solutions of the Example 3 with $\varrho_1(x) = e^x - x^2$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$

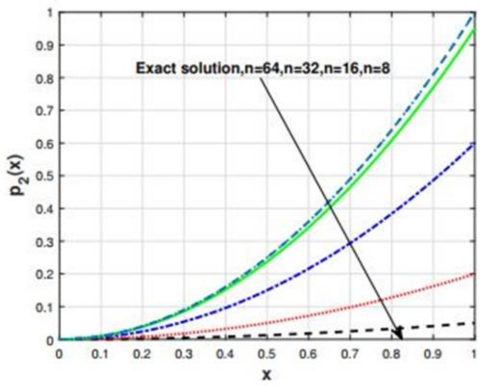

Fig. 6. Numerical and exact solutions of the Example 3 with $\varrho_2(x) = \sin^2(x)$ and for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$

Table 5. The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_1(x) = e^x - x^2$ for Example 3

-34-5 x	$\|p_1(x) - p_{1,n}(x)\|$		$\|p_1(x) - p_{1,n}(x)\|$	
	n=8	n=16	n=32	n=64
0.1	$1.35242e-15$	$1.42118e-16$	$1.49223e-17$	$1.56560e-18$
0.2	$1.09957e-15$	$1.15953e-16$	$1.22164e-17$	$1.28592e-18$
0.3	$0.88038e-15$	$0.93215e-16$	$0.98592e-17$	$1.04171e-18$
0.4	$0.69244e-15$	$0.73662e-16$	$0.78265e-17$	$0.83056e-18$
0.5	$0.53335e-15$	$0.57055e-16$	$0.60944e-17$	$0.65006e-18$
0.6	$0.40071e-15$	$0.43153e-16$	$0.46388e-17$	$0.49781e-18$
0.7	$0.29212e-15$	$0.31715e-16$	$0.34356e-17$	$0.37141e-18$
0.8	$0.20516e-15$	$0.22501e-16$	$0.24609e-17$	$0.26845e-18$
0.9	$0.13744e-15$	$0.15270e-16$	$0.16905e-17$	$0.18652e-18$

Table 6. The absolute error for the values of the parameters $n, \mu = \frac{1}{2}, \nu = \frac{1}{4}$ and $\varrho_2(x) = \sin^2(x)$ for Example 3

-34-5 x	$\|p_2(x) - p_{2,n}(x)\|$		$\|p_2(x) - p_{2,n}(x)\|$	
	n=8	n=16	n=32	n=64
0.1	$1.96991e-15$	$2.02502e-16$	$2.08087e-17$	$2.13749e-18$
0.2	$1.75711e-15$	$1.80917e-16$	$1.86199e-17$	$1.91558e-18$
0.3	$1.55647e-15$	$1.60549e-16$	$1.65527e-17$	$1.70581e-18$
0.4	$1.36799e-15$	$1.41398e-16$	$1.46072e-17$	$1.50822e-18$
0.5	$1.19168e-15$	$1.23462e-16$	$1.27832e-17$	$1.32277e-18$
0.6	$1.02752e-15$	$1.06742e-16$	$1.10808e-17$	$1.14950e-18$
0.7	$0.87552e-15$	$0.91238e-16$	$0.95000e-17$	$0.98838e-18$
0.8	$0.73568e-15$	$0.76950e-16$	$0.80408e-17$	$0.83942e-18$
0.9	$0.60800e-15$	$0.63878e-16$	$0.67032e-17$	$0.70262e-18$

4.3. Example 3

We describe the following nonlinear coupled differential-integral equation with $\varrho_1(x) = e^x - x^2, \varrho_2(x) = \sin^2(x)$:

$$ {}^{Caputo}D_{\varrho_1(x)}[p_1(x)] + p_1{}'(x) + \int_0^x p_1(v)dv = 2x^2 + \frac{x^4}{12} + \frac{x^9}{27} - p_1^3(x), $$
$$ p_1(0) = p_1{}'(0) = 0, \quad (49) $$
$$ {}^{Caputo}D_{\varrho_2(x)}[p_2(x)] + p_2{}'(x) + \int_0^x p_2(v)dv = 4x + \frac{7x^3}{3} - p'_2(x)p_2(x), $$
$$ p_2(0) = p_2{}'(0) = 0, \quad (50) $$

the exact solutions of Eqs.(49),(50) are $p_1(x) = \frac{1}{3}x^3, p_2(x) = x^2$. This Example is solved by the proposed method and the results of the approximate and exact solutions are shown in the Figs. 5 and 6. In the Tables 5 and 6, the absolute error between the approximate and analytical solutions are displayed.

5. Conclusions

This article is focused on the coupled differential-integral equation including the Caputo fractional operator of variable-orders and it is proposed a numerical method based on the shifted Jacobi - Gauss collocation scheme to obtain the solution of the coupled differential-integral equation. Also, in this paper about the convergence and an upper bound on the error are discussed. Some numerical Examples have been showed in order to display the high exactness of the suggested method.

Conflicts of Interest

The authors declare no conflict of interest.

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