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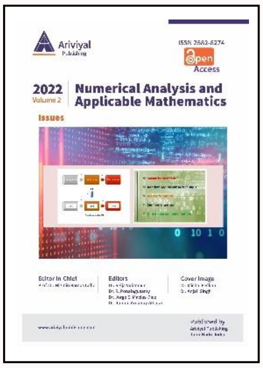

A Numerical Scheme for Solving Convolution Type Volterra Integral Equations

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Abstract: In this paper a numerical scheme is constructed for approximation of Volterra integral equation of convolution type. The numerical scheme is based on Laplace transform and quadrature. By the use of Laplace transform the integral equation is converted to a system of equations in the transformed space and then inverse Laplace is numerically approximated in the complex plane along hyperbolic contour. The numerical scheme and its error estimate is discussed and validated for various VIE problems.

Keywords: Volterra; Integral Equation (VIE); Laplace Transform; Quadrature Method; Trapezoidal Rule; Hyperbolic Contour

1. Introduction

The Volterra integral equations have many important applications in many fields of sciences like mathematics and engineering. VIE can be utilized to model many physical systems in continuum mechanics of material, population dynamics, financial mathematics, spread of epidemics, fluid dynamics, diffusion problems and electrostatics problems.[1-22] etc. Many efficient methods have been developed over the years to approximate the Volterra integral equations.[2-9] More detail can be found in the book.[22] In the present work the Laplace transform based method like[13,17] is extended to solve Volterra integral equations of the following form

$$y(t) = \int_0^t K(t-s)y(s)ds + f(t) \tag{1}$$

where K denotes kernel function of convolution type, and $f(t)$ is given function.

2. Description of the method for VIE

By applying the Laplace transforms to equation (1), and let $Y(s)$ be the solution of the transform problem. The solution of VIE in equation (1) can be computed by applying the inverse Laplace transform to get the following

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(s)e^{st}ds \tag{2}$$

A contour can be used to approximate the line $c - i\infty to\ c + i\infty$, for example parabolic or hyperbolic. The above integrand will beexponentially decayed $if c - i\infty\ to\ c + i\infty$ can be deformed into the left half plane. In such a case the parametric equation of hyperbola due to[17] is defined by the following equation,

$$s = \omega + \lambda(1 - \sin(\sigma - i\mu)), -\infty < \mu < \infty \qquad (\Gamma) \tag{3}$$

Table 1. Comparison of the solutions for various values of quadrature nodes N of the present numerical scheme at time $t = 1$ corresponding to problem 1.

N	E	$\in$
5	0.2466	0.0447
10	0.0182	0.0130
20	0.0012	0.0013
40	$2.1061e^{-006}$	$1.9533e^{-005}$
80	$1.6249e^{-010}$	$1.1786e^{-008}$
100	$3.8858e^{-013}$	$3.7104e^{-010}$
150	$2.2259e^{-015}$	$9.9729e^{-014}$
200	$3.1142e^{-015}$	$4.0395e^{-017}$
.[16]	$3.120e^{-002}$	

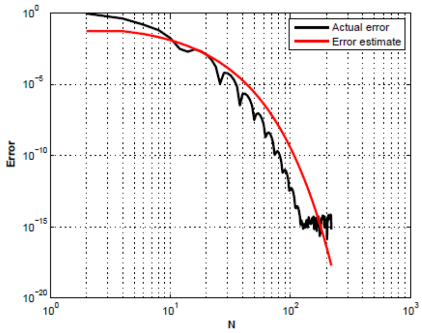

Fig. 1. Actual error versus quadrature nodes N at time $t = 1$ corresponding to problem 1.

Γ represents left branch of hyperbola and is defined by the following equation

$$\left(\frac{x-\omega-\lambda}{\lambda \sin \sigma}\right)^2 - \left(\frac{y}{\lambda \cos \sigma}\right)^2 = 1 \tag{4}$$

So the solution of problem (1) defined in equation (2) is now reduced to the following form,

$$y(t) = \frac{1}{2\pi i} \int_{\Gamma} Y(s) e^{st} ds \tag{5}$$

Where the Γ denotes the parametric form of the path defined in equation (3) we write $s = s(\mu)$ the above equation (5) reduced to the following form,

$$y(t) = \frac{1}{2\pi i} \int_{\Gamma} Y\big(s(\mu)\big) e^{s(\mu)t} s'(\mu) d\mu \tag{6}$$

Where $Y(s(\mu)) = Y(s)$ and using equal weight quadrature rule with $K > 0, s_j = s(\mu_j), s'_j = s'(u_j)$, the equation (6) can be approximated by

$$y_N(t) = \frac{K}{2\pi i} \sum_{j=-N}^{N} Y(s_j) e^{s_j(t)} s_j' \tag{7}$$

3. Numerical Experiments

The present numerical scheme is now applied for solving various types of Volterra integral equations linear as well as nonlinear of convolution type. All the numerical result obtained using the following values of optimal parameters derived by the authors in McLean W et al.,[17] given by $t \in [t_0 = 0.01, T = 5]$, $\theta = 0.1$, $\sigma = 0.3812$, $\tau = \left(\frac{t_0}{T}\right)$, $b = \cos h^{-1}\left(\frac{1}{\theta\tau \sin(\sigma)}\right)$, $r = 0.3431$, $r' = 2\pi r$, $K = \frac{b}{N}$, $\omega = 0.2$, $\mu = r'\left(\frac{(1-\theta)}{b}\right)$, $\lambda = \frac{\theta r' N}{bT}$, $\rho(r) = \frac{\theta r' \tau \sin(\sigma - r)}{b}$, $l(\rho_r N) = \max(1, \log(\frac{\frac{1}{N}}{\rho_r}))$, $r' = 2\pi r \in \cong l(\rho_r N) e^{-\mu N}$.

3.1. Problem

$$y(t) = t + \frac{4}{3} t^{\frac{3}{2}} - \int_0^t \frac{y(x)}{\sqrt{t-x}} dx \tag{8}$$

It is easily to show that the Laplace transform of equation (8) is$(s) = 1/s^2$, which can be used in equation (7) to obtain the numerical solution y_N for different values of quadrature points N. The accuracy and performance of the present numerical scheme is tested in term of actual error E and compared with the error estimate $\in$ of the numerical scheme at time $t = 1$ where the exact solution is $y(t) = t$, Babolian E.,[16] the hyperbolic path is used with its optimal values discussed above. It can be seen from Table 1 and Fig. 1 that the present method is highly accurate as compared to the methods discussed[16] for this problem.

3.2. Problem

$$y(t) = \int_0^t e^{(t+x)} y(x)\, dx = te^t \tag{9}$$

The Leibnitz generalized formula (see for example[18]) can be used to reduce the problem (3) to form

Table 2. Numerical solution: comparison of actual error of the present scheme, its error estimate for various values of quadrature nodes, at time $t = 1$ corresponding to problem 2.

N	E	$\in$
5	0.4577	0.0447
10	0.0410	0.0130
20	$7.1677e^{-004}$	0.0013
40	$1.7981e^{-007}$	$1.9533e^{-005}$
80	$5.1235e^{-014}$	$1.1786e^{-008}$
100	$4.1003e^{-015}$	$3.7104e^{-010}$
150	$9.1056e^{-015}$	$9.9729e^{-014}$
200	$1.4103e^{-014}$	$4.0395e^{-017}$
.[15]	$9.310e^{-004}$	
.[19]	$4.700\ e^{-006}$	

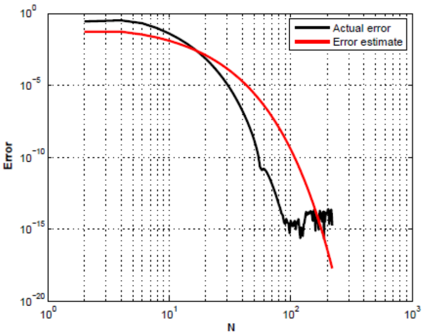

Fig. 2. Actual error versus quadrature nodes N at time $t = 1$ corresponding to problem 2.

Table 3. Numerical solution: comparison of actual error of the present scheme, its error estimate for various values of quadrature nodes, at time $t = 1$ corresponding to problem 3.

N	E	$\in$
5	0.1292	0.0447
10	0.0129	0.0130
20	$7.3658e^{-005}$	0.0013
40	$6.2040e^{-006}$	$1.9533e^{-005}$
80	$1.0153e^{-010}$	$1.1786e^{-008}$
100	$4.5356e^{-013}$	$3.7104e^{-010}$
150	$1.2707e^{-013}$	$9.9729e^{-014}$
200	$2.9845e^{-013}$	$4.0395e^{-017}$
.[16]	$3.280e^{-002}$	

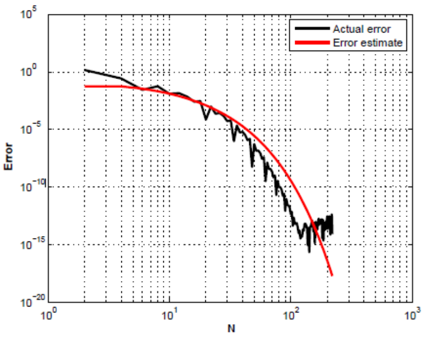

Fig. 3. Actual error versus quadrature nodes N at time $t = 1$ corresponding to problem 3.

$$y(t) + \int_0^t e^{(x-t)} y(x)\, dx = e^{-t}(1+t) \tag{10}$$

Applying the Laplace transform to equation (10) we get

$Y(s) = \frac{1}{(1+s)}$ This transformed value of the solution $y(t)$ can be used in (7) to get the numerical solution of problem (10). The exact solution of this problem is$(t) = e^{-t}$. The actual error E, error estimate $\in$ of the numerical scheme are shown in Table 2 and Fig. 2 for different choices of quadrature nodes N and time $t = 1$the results of the present scheme are compared with the results for the methods in.[15,19]

3.3. Problem

Consider the nonlinear VIE.[16]

$$e^{2t} - e^t = \int_0^t e^{(t-x)}\, y^2(x) dx \tag{11}$$

let us denote $y^2\ (x) = v(x)$, then the equation (11) reduces to

$$e^{2t} - e^t = \int_0^t e^{(t-x)}\, v(x) dx \tag{12}$$

by applying the Laplace transform to equation (12) we have the following value of the transformed solution,

$$V(s) = \frac{1}{s-2} \tag{13}$$

$$y_N(t) = \sqrt{v_N}(t) \tag{14}$$

Where $y_N(t)$ can be computed from equation (7) where the exact solution of the problem (11) is $\mu(t) = e^t$the actual solution error E and the estimated errors $\in$ are shown in Table 3 and Fig. 3. The results of the present method are compared with the.[16] at time $t = \frac{1}{8}$ for this particular problem, it can be seen this numerical method is highly accurate.

3.4. Problem

Consider the nonlinear VIE.[10]

$$\int_0^t e^{(t-x)} \ln\big(y(x)\big) dx = e^t - t - 1 \tag{15}$$

Table 4. Numerical solution: comparison of actual error of the present scheme, its error estimate for various values of quadrature nodes, at time $t = 1$ corresponding to problem 4.

N	E	$\in$
5	0.4569	0.0447
10	0.0266	0.0130
20	0.0016	0.0013
40	$4.4580e^{-011}$	$1.1786e^{-008}$
80	$2.4053e^{-013}$	$3.7104e^{-010}$
100	$6.956e^{-016}$	$3.7104e^{-010}$
150	$1.9050e^{-013}$	$9.9729e^{-014}$
200	$45324.e^{-013}$	$4.0395e^{-017}$

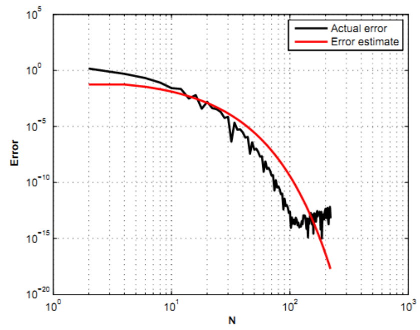

Fig. 4. Actual error versus quadrature nodes N at time t=1 corresponding to problem 4.

Let $\ln(y(x)) = v(x)$, the equation (15) become,

$$\int_0^t e^{(t-x)}\, \mathrm{v}(\mathrm{x})\, dx = e^t - t - 1 \quad (16)$$

taking Laplace of equation (16) we have,

$$V(s) = \frac{1}{s^2} \quad (17)$$

$$y_N(t) = e^{v_N}(t) \quad (18)$$

where $v_N(t)$ can be computed using equation (7) and the exact solution is given as $\mu(t) = e^t$. This problem is solved by the present Laplace transform based method and the results are shown in Table 4 and Fig. 4. The actual error E, the error estimate $\in$ is computed for different values of quadrature nodes N. It can be observed that for large range of quadrature nodes the actual error and the error estimate well agreed.

4. Conclusions

In this present work a numerical scheme is implemented for solving the VIE which is based on Laplace transform. The numerical coupled Laplace transform with quadrature rule and the resultant numerical scheme highly accurate and efficient for the numerical solution of VIE. The numerical scheme is compared with other available methods and it is found that the present method having more accuracy than the large range interpolation, integral expansion, operational matrix with block-pulse function and piecewise constant orthogonal function, and optimal homotopy asymptotic method. The method is applicable to Volterra integral equations of convolution type.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1 Khater A.H.; Shamardan A.B.; Callebaut D.K.; Sakran M.R.A. Numerical Solutions of Integral and Integro-Differential Equations using Legendre Polynomials. *Numer. Algorithms*, 2007, **46**, 195-218. [CrossRef]
2 Masouri Z.; Babolian E.; Hatamzadeh-Varmazyar S. An Expansion–Iterative Method for Numerically Solving Volterra Integral Equation of the First Kind. *Comput. Math. Appl*., 2010, **59**, 1491-1499. [CrossRef]
3 Grundy R.E. The Solution of Volterra Integral Equations of the Convolution Type using Two-Point Rational Approximants. *IMA J. Appl. Math*., 1978, **22**, 147-158. [CrossRef]
4 Olmstead M.L.; Nicholson R.S. A Method Based on Polynomial Approximations for Numerical Solution of Volterra Integral Equations. *J. Electroanal. Chem. Interfacial Electrochem*., 1968, **16**, 145-151. [CrossRef]
5 Conte D.; Del Prete I. Fast Collocation Methods for Volterra Integral Equations of Convolution Type. *J. Comput. Appl. Math*., 2006, **196**, 652-663. [CrossRef]
6 Bushell P.J.; Okrasinski W. Nonlinear Volterra Integral Equations with Convolution Kernel. *J. Lond. Math. Soc*., 1990, **2**, 503-510. [CrossRef]

7 Fagnani F.; Pandolfi L. A Recursive Algorithm for the Approximate Solution of Volterra Integral Equations of the First Kind of Convolution Type. *Inverse Probl*., 2002, **19**, 23. [Link]
8 Keilson J.; Nunn W.R. Laguerre Transformation as a Tool for the Numerical Solution of Integral Equations of Convolution Type. *Appl. Math. Comput*., 1979, **5**, 313-359. [CrossRef]
9 Maleknejad K.; Aghazadeh N. Numerical Solution of Volterra Integral Equations of the Second Kind with Convolution Kernel by using Taylor-Series Expansion Method. *Appl. Math. Comput.*, 2005, **161**, 915-922. [CrossRef]
10 Saeedi L.; Tari A.; Momeni Masuleh S.H. Numerical Solution of Some Nonlinear Volterra Integral Equations of the First Kind. *Applications and Applied Mathematics: An International Journal (AAM)*, 2013, **8**, 13. [Link]
11 Rashed M.T. An Expansion Method to Treat Integral Equations. *Appl. Math. Comput*., 2003, **135**, 65-72. [CrossRef]
12 Rashed M.T. Lagrange Interpolation to Compute the Numerical Solutions of Differential, Integral and Integro-Differential Equations. *Appl. Math. Comput*., 2004, **151**, 869-878. [CrossRef]
13 Weideman J.; Trefethen L. Parabolic and Hyperbolic Contours for Computing the Bromwich Integral. *Math. Comput*., 2007, **76**, 1341-1356. [Link]
14 McLean W.; Thomee V. Time Discretization of an Evolution Equation via Laplace Transforms. *IMA J. Numer. Anal*., 2004, **24**, 439-463. [CrossRef]
15 Babolian E.; Masouri Z. Direct Method to Solve Volterra Integral Equation of the First Kind using Operational Matrix with Block-Pulse Functions. *J. Comput. Appl. Math*., 2008, **220**, 51-57. [CrossRef]
16 Babolian E.; Shamloo A.S. Numerical Solution of Volterra Integral and Integro-Differential Equations of Convolution Type by using Operational Matrices of Piecewise Constant Orthogonal Functions. *J. Comput. Appl. Math*., 2008, **214**, 495-508. [CrossRef]
17 McLean W.; Thomee V. Numerical Solution via Laplace Transforms of a Fractional Order Evolution Equation. *J. Integral Equ. Appl*., 2010, **22**, 57-94. [Link]
18 Wazwaz A.M. First Course *In Integral Equations, A*. World Scientific Publishing Company, 2015. [Link]
19 Khan N.; Hashmi M.S.; Iqbal S.; Mahmood T. Optimal Homotopy Asymptotic Method for Solving Volterra Integral Equation of First Kind. *Alex. Eng. J*., 2014, **53**, 751-755. [CrossRef]
20 Ding H.J.; Wang H.M.; Chen W.Q. Analytical Solution for the Electroelastic Dynamics of a Nonhomogeneous Spherically Isotropic Piezoelectric Hollow Sphere. *Arch. Appl. Mech*., 2003, **73**, 49-62. [CrossRef]
21 Baratella P. A Nystrom Interpolant for Some Weakly Singular Linear Volterra Integral Equations. *J. Comput. Appl. Math*., 2009, **231**, 725-734. [CrossRef]
22 Polyanin A.D.; Manzhirov A.V. *Handbook of Integral Equations*. Chapman and Hall/CRC, 2008. [CrossRef]

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