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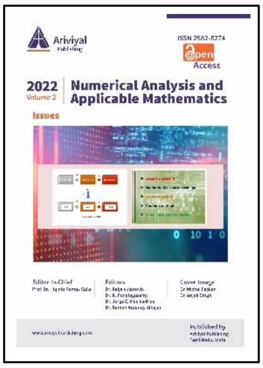


# On a New Subclass of Univalent Functions Defined by Salagean Differential Operator

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**Abstract:** The purpose of writing this paper is to get the geometric properties of new subclasses of analytic and univalent functions defined by Salagean differential operator.

**Keywords:** Analytic and univalent functions; Salagean differential operator; Coefficient estimates; Fekete-Szego functional; Inclusion relation

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## 1. Introduction

Let A denotes the class of functions analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$; $f(z)$ is said to be in the class $S$ (subclass of A) if $f(z)$ is analytic and univalent in the unit disk $\mathbb{U}$ which can be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

satisfying the conditions $f(0) = 0$; $f'(0) = 1$. Class $S$ is called the class of normalized univalent functions.

In 1851, Riemann[1] provided one of the most classical and remarkable results in geometric function theory which is the well-known Riemann mapping theorem and it states that "there exists a unique analytic functions which maps a simply connected domain onto another simply connected domain in a complex plane”. In view of Riemann mapping theorem, the study of the properties of analytic and univalent functions began by Koebe,[2] in 1907 and he provided the first important paper in this area, followed by Alexander[3] and Bieberbach[4] in 1915 and 1916 respectively. In 1985, the Bieberbach conjecture which is also known as coefficient conjecture was proved by Branges[5] which states that “for a univalent function, it's nth-Taylor's coefficient is bounded by n”.

So many researchers, have defined different classes of analytic univalent, multi-valent and bi-univalent functions and obtained their coefficient estimates. Fekete-Szego inequality is an inequality involving the coefficients of univalent analytic functions found by Fekete and Szego,[6] which is related to Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete-Szego problem. The Fekete-Szego inequality states that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

is a univalent analytic function on the unit disk $\mathbb{U}$ and $0 \le \lambda \le 1$; then

$$C(\lambda) = |a_3 - \lambda a_2^2| < 1 + 2\exp(-2\lambda)/(1-\lambda))$$

For a class of function in A and a real number $\mu$, the Fekete-Szego problem is all about finding the best possible upper bound for $C(\lambda)$ for functions in A. A function f $\in$ S of the form (1.1) is star-like in the unit disk $\mathbb{U}$ if it maps a unit disk onto a star-like domain. A necessary and sufficient condition for f to be star-like is that:

$$Re\left\{z\frac{f'(z)}{f(z)}\right\} > 0, z \in \mathbb{U}$$

The class of all-star-like functions is denoted by $S^*$

Let $P$ be the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} C_n z^n \tag{1.2}$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfies the conditions $Re(p(z)) > 0$ and $p(z) = 1$, for $z \in \mathbb{U}$, $p(z)$ is called a Caratheodory function or function having positive real part in $\mathbb{U}$.

An analytic function $f(z)$ is convex if it maps $\mathbb{U}$ conformally onto a convex domain. Equivalently, $f \in C$ if and only of it satisfies the following condition;

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, z \in \mathbb{U}.$$

The class of all convex functions is denoted by $K$ And if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in \mathbb{U}$$

$0 \leq \alpha \leq 1$, $f(z)$ is a convex function of order $\alpha$. The class of all convex functions of order $\alpha$ is defined as $f(\alpha)$.
Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{U}$, then $f(z)$ is subordinate to $g(z)$ written as $f(z) \prec g(z)$, if there exist a function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ which is called the Schwarz function such that $f(z) = g(\omega(z))$.
If the function $g$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z), z \in \mathbb{U} \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.[7]

**Maximum Modulus Theorem**: Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic. If there exists a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$, for all $z \in D$, then $f$ is constant on $D$.

**Lemma 1.1.** ([8]) If $p \in P$ with the series expansion $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, $z \in \mathbb{U}$ with $Re(p(z)) > 0$ and $(p(0)) = 1$. Then $|p_n| \leq 2$, $n = \{1,2, \dots\}$ and the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$, $z \in \mathbb{U}$

**Lemma 1.2.** ([9]) Let the function $p \in P$ be given by equation (1.2), then for any complex number $\sigma$

$$|p_2 - \sigma p_1^2| \leq 2max\{1, |2\sigma - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z}{1-z} \cdot z \in \mathbb{U}$$

Motivated by the work of Krishna et al[10] the following definitions are given.

**Definition**: A function $f(z) \in A$ is said to be in the class $S_n(\alpha, \beta, \gamma)$ if it satisfies the condition that

$$\mathbb{R}\left\{\alpha\frac{D^n f(z)}{z} + \beta(D^n f(z))'\right\} > \gamma, z \in \mathbb{U} \tag{1.3}$$

where $\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$, $n = \{0\} \cup \{1,2,3 \dots\}$. $D^n f(z)$ is the well known Salagean differential operator and it is defined as[11]

$$D^n f(z) = z + \sum_{k=2}^{\infty} K^n a_k z^k$$

**Definition**: For $-1 \leq B < A \leq 1$, $n \in \mathbb{N} \cup \{0\}$. A function $f \in A$ is in the class $R_n(A, B)$ if

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U} \tag{1.4}$$

## 2. Main results

Next is to obtain the geometric properties of the new classes of functions defined, such as coefficient estimates, Fekete-Szego functional and the inclusion relation.

## 3. Coefficient Estimates

**Theorem:** A function ***f*** of the form (1.1) belongs to the class $S_n(\alpha,\beta,\gamma)$ if

$$a_k = \frac{(\alpha+\beta-\gamma)C_k-1}{k^n(\alpha+\beta k)}, k \geq 2$$

**Proof**: Let $f \in S_n(\alpha,\beta,\gamma)$, there exists an analytic functions $p \in P$ in the open unit disk $\mathbb{U}$ satisfying the conditions $p(0)=0, Re(p(z))>0$ such that

$$\frac{1}{\alpha+\beta-\gamma}\left\{\alpha\frac{D^n f(z)}{z}+\beta(D^n f(z))'-\gamma\right\}=p(z)$$

which implies that

$$\left\{\alpha\frac{D^n f(z)}{z}+\beta(D^n f(z))'-\gamma\right\}=(\alpha+\beta-\gamma)\,(p(z))$$

on expansion of the last expression, the result is;

$$a_2 z(\alpha 2^n+\beta 2^{n+1})+a_3 z^2(\alpha 3^n+\beta 3^{n+1})+a_4 z \quad (\alpha 4^n+\beta 4^{n+1})+a_5 z^4(\alpha 5^n+\beta 5^{n+1})+\cdots. \tag{3.1}$$

$$=(\alpha+\beta-\gamma)(c_1 z+c_2 z^2+c_3 z^3+c_4 z^4+\cdots) \tag{3.2}$$

comparing the like powers of the coefficients of $z$ on both sides of equations (3.1) and (3.2),

$$a_2=\frac{(\alpha+\beta-\gamma)c_1}{2^n(\alpha+2\beta)}$$
$$a_3=\frac{(\alpha+\beta-\gamma)c_2}{3^n(\alpha+3\beta)}$$
$$a_4=\frac{(\alpha+\beta-\gamma)c_3}{4^n(\alpha+4\beta)}$$
$$a_5=\frac{(\alpha+\beta-\gamma)c_4}{5^n(\alpha+5\beta)}$$

In general,

$$a_k = \frac{(\alpha+\beta-\gamma)C_k-1}{k^n(\alpha+\beta k)}, k \geq 2 \tag{3.3}$$

**Remark 2**

For $n=0$, $a_k$ equation (3.3) reduces to the result obtained in Krishna et al.[10]

## 4. Fekete-Szego Functional

**Theorem:** If $f(z)$ given by equation (1.1) belongs to the class $S_n(\alpha,\beta,\gamma)$, then for any complex number $\sigma$

$$|a_3-\sigma a_2^2|=\frac{(\alpha+\beta-\gamma)}{3^n(\alpha+3\beta)}max\left\{1,\left|\frac{3^n\sigma(\alpha+3\beta)}{2^{2n}(\alpha+2\beta)^2}\right|\right\}$$

$\alpha,\beta>0$ and $0\leq\gamma<\alpha+\beta\leq 1, n=\{0\}\cup\{1,2,3\ldots\}$

**Proof:** from equation (3.3),

$$a_2=\frac{(\alpha+\beta-\gamma)c_1}{2^n(\alpha+2\beta)}$$
$$a_3=\frac{(\alpha+\beta-\gamma)c_2}{3^n(\alpha+3\beta)}$$

then the Fekete-Szego functional can be written as

$$|a_3-\sigma a_2^2|=\frac{(\alpha+\beta-\gamma)}{3^n(\alpha+3\beta)}\left\{c_2-\frac{3^n\sigma c_1^2(\alpha+3\beta)}{2^{2n}(\alpha+2\beta)^2}\right\} \tag{4.1}$$
$$=\frac{(\alpha+\beta-\gamma)}{3^n(\alpha+3\beta)}\{c_2-\upsilon c_1^2\} \tag{4.2}$$

Where

$$\upsilon=\frac{3^n\sigma(\alpha+3\beta)}{2^{2n}(\alpha+2\beta)^2}$$

Applying lemma (1.2)

$$|a_3 - \sigma a_2^2| = \frac{(\alpha+\beta-\gamma)}{3^n(\alpha+3\beta)} max\left\{1, \left|\frac{3^n\sigma(\alpha+3\beta)}{2^{2n}(\alpha+2\beta)^2}\right|\right\}$$

and the result follows.

**Corollary**: when $n = 0$, the following corollary is obtained;

$$|a_3 - \sigma a_2^2| = \frac{(\alpha+\beta-\gamma)}{(\alpha+3\beta)} max\left\{1, \left|\frac{\sigma(\alpha+3\beta)}{(\alpha+2\beta)^2}\right|\right\}$$

## 5. Inclusion Relation

Theorem: A function $f \in A$ belongs to the class $R_n(A, B)$ if and only if the following condition is satisfied

$$\sum_{k=2}^{\infty}\{(1+B)k - (1+A)\}k^n a_k \leq A - B \tag{5.1}$$

$-1 \leq B < A \leq 1, n \in \mathbb{N} \cup \{0\}$.

Proof: Let $f \in R_n(A, B)$. If

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec \frac{1+Az}{1+Bz}, z \in \mathbb{U}$$

then by the definition of subordination, there exists $w(z) \in A$ such that $|w(z)| < 1, w(0) = 0$ and equation (1.4) can be written as

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec \frac{1+Aw(z)}{1+Bw(z)}$$

which implies that

$$\big(1 + Aw(z)\big)D^n f(z) = (1 + Bw(z))z(D^n f(z))'$$

$$w(z) = \frac{z(D^n f(z))' - D^n f(z)}{AD^n f(z) - B(zD^n f(z))'}$$

$$|w(z)| = \left|\frac{z\big(D^n f(z)\big) - D^n f(z)}{AD^n f(z) - Bz(D^n f(z))'}\right|$$

writing $|w(z)|$ in the form of power series;

$$|w(z)| = \left|\frac{\sum_{k=2}^{\infty} k^n a_k z^k\,(k-1)}{(A-B)z + \sum_{k=2}^{\infty} k^n a_k z^k (A - Bk)}\right|$$

it can be recalled that $|w(z)| < 1$ from the definition of subordination. Hence,

$$|w(z)| = \left|\frac{\sum_{k=2}^{\infty} k^n a_k z^k\,(k-1)}{(A-B)z + \sum_{k=2}^{\infty} k^n a_k z^k (A - Bk)}\right| < 1$$

Since $Re(z) \leq |z| < 1$ and taking the value of $|z| \to 1$ on the real axis,

$$\sum_{k=2}^{\infty}\{(1+B)k - (1+A)\}\,k^n a_k \leq A - B$$

Conversely, suppose

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k$$

And

$$\sum_{k=2}^{\infty}\{(1+B)k - (1+A)\}\,k^n a_k \leq A - B$$

it suffices to show that

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec \frac{1+Aw(z)}{1+Bw(z)}, z \in \mathbb{U}$$

i.e. $|w(z)| < 1$

$$|w(z)| = \left|\frac{\sum_{k=2}^{\infty} k^n a_k z^k\,(k-1)}{(A-B)z + \sum_{k=2}^{\infty} k^n a_k z^k (A - Bk)}\right| < 1$$

$$\left|\sum_{k=2}^{\infty} k^n a_k z^k (k-1)\right| < \left|(A-B)z + \sum_{k=2}^{\infty} k^n a_k z^k (A-Bk)\right| < 0$$

$$\left|\sum_{k=2}^{\infty} k^n a_k z^k (k-1)\right| - \left|(A-B)z + \sum_{k=2}^{\infty} k^n a_k z^k (A-Bk)\right|$$

$$\leq \left|\sum_{k=2}^{\infty} k^n a_k z^k (k-1) - (A-B)z - \sum_{k=2}^{\infty} k^n a_k z^k (A-Bk)\right|$$

Taking the value of $|z| \to 1$ on the real axis

$$\left|\sum_{k=2}^{\infty} k^n a_k z^k (k-1) - (A-B)z - \sum_{k=2}^{\infty} k^n a_k z^k (A-Bk)\right|$$

$$\leq \sum_{k=2}^{\infty} k^n \{(1+B)k - (1+A)\} a_k - (A-B) \leq 0$$

Hence, by maximum modulus theorem, $f \in R_n(A,B)$.

## 6. Conclusions

Salagean differential operator was used to define two new subclasses of analytic and univalent functions in the unit disk. The geometric properties (such as the coefficient estimates, Fekete-Szego functionals and the inclusion relation) of the new subclasses defined were studied, the results obtained is an improvement on the existing results and new results were also obtained.

## Conflicts of Interest

The authors declare no conflict of interest.

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