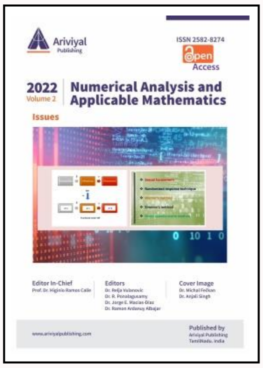
Numerical Analysis and Applicable Mathematics

Research Article

DOI:10.36686/Ariviyal.NAAM.2022.03.06.016

Numer. Anal. Appl. Math., 2022, 3(6), 39-50.



Numerical Solution of Stochastic Delay Differential Equations Using Block Backward Differentiation Formulae Methods

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ISSN: 2582-8274

Crossref

Publication details
Received: 19th June 2022
Revised: 03rd December 2022
Accepted: 03rd December 2022
Published: 20th December 2022

Abstract: This paper presents the application of Block Backward Differentiation Formulae Methods for the numerical solution of Stochastic Delay Differential Equations with two new formulated expressions for evaluation of the delay terms. This was carried-out by formulating a continuous representation of the proposed method through multistep collocation method by matrix inversion technique. After the evaluation and simplification of the continuous representations of each step number, the discrete schemes were obtained. The convergence and stability analysis of the method were investigated. The performances of the method were demonstrated by solving some stochastic delay differential equations to show the accuracy and efficiency advantages over other existing methods. It was observed that the scheme for step number k = 4 performed slightly better and faster in terms of accuracy than the schemes for step number k = 2 and 3 respectively when compared with the exact solutions.

Keywords: Stochastic delay differential equations; block method; linear multistep method

Mathematics Subject Classification 2010: 34K28; 65F30; 90C30; 90C26.

1. Introduction

Stochastic delay differential equation (SDDE) is a stochastic differential equation where the increment of the process depends not only on current state but also on the history part which contains the random values of the system being modeled. The applications of SDDEs can be seen in applied sciences, economics and engineering. Most scholars such as[1-6] applied Euler-Maruyama scheme to develop a continuous split-step scheme of SDDE on a continuous interval $t_0 \leq t \leq t_a$ in order to obtain its numerical solutions and used interpolation techniques in evaluating the delay term but encountered some setbacks. One of the setbacks encountered by these scholars in the application of the interpolation techniques such as Hermite, Nordsieck, and Newton divided difference and Neville's interpolation to evaluate the delay term was studied by[7] that the order of the interpolating polynomials should be at least the same with numerical methods which is very difficult to carry-out. Researchers[8-15] applied the formula developed by[16] for the evaluation of the delay term of delay differential equations and discovered that it gives lesser accurate results, it takes more time to compute and cannot be adequately use in solving different classes of DDEs with multiple delay terms. It is highly required that an accurate mathematical expression need to be develop to address these observations discovered. In this research work, we shall be constructing and applying Block Backward Differentiation Formulae Methods (BBDFMs) as a linear multistep collocation method to discretize SDDEs on a discrete interval (t_0, t_a) in order to obtain its discrete schemes from the continuous representations of each step number through matrix inversion approach. These discrete schemes obtained shall be applied in solving some SDDEs with accurate and efficient formulae in evaluating the delay terms which gives more accurate results, lesser time to compute and can also solve different classifications of DDEs.
From,[17] stochastic delay differential equation (SDDE) can be express as

$$\begin{aligned}&dX(t) = f(X(t), X(t-\tau), t)dt + g(X(t), X(t-\tau), t)dW(t) \ \text{ for } t > t_0, \tau > 0\\ &X(t) = \xi(t), \text{ for } t \leq t_0\end{aligned} \tag{1}$$

where $\xi(t)$ is the initial function, $X(t)$ is the stochastic process of the current state, τ is called the delay, $(t-\tau)$ is called the delay term and $X(t-\tau)$ is the solution of the delay term on the drift part and $W(t)$ is the standard Brownian motion with its differential equivalence as $dW(t)$ which is the noise term or Wiener process together with solution of the delay term as $X(t-\tau)dW(t)$ on the diffusion part of (1).

1.1. Existence and Uniqueness of Solutions

Here, we shall state the theorem for existence and uniqueness solutions for equation (1) as in [18].

Theorem 1: Let the current state and history part of the drift and diffusion coefficients of (1) be represented as a and b, then the functions $f(a,b,t)$ and $g(a,b,t)$ satisfying the condition $\| f(a_1,b_1,t)+g(a_2,b_2,t) \| \leq S(t)+R(t)(\| a_1+a_2 \| + \| b_1+b_2 \|)$ in $(t_0,t_z)\times\Re^e\times\Re^e$, where $S(t)$ and $R(t)$ are continuous positive functions on (t_0,t_z), then the solution of (1) exist and is unique on the entire discrete interval (t_0,t_z). Taking into consideration the sequence of points $\{t_z\}$ defined by $t_z=t_0+zl$, $z=1,2,\ldots$. where the parameter l is the stepsize, the computational solution is sought on the discrete point set $\left\{t_z | z=1,2,\ldots,\frac{t_z-t_0}{l}\right\}$ and not on continuous interval $t_0\leq t\leq t_z$.

2. Method of Formulation

2.1. Formulation of Multistep Collocation Method

In [19], a *k*-step multistep collocation method with *q* collocation points was formulated as

$$y(x)=\sum_{u=0}^{p-1}\phi_u(x)y_{z+u}+l\sum_{u=0}^{q-1}\psi_u(x)\,f(x,y(x)) \tag{2}$$

where $\phi_u(x)$ and $\psi_u(x)$ are continuous coefficients of the method defined as

$$\phi_u(x)=\sum_{v=0}^{p+q-1}\phi_{u,v+1}\,x^v \quad for\ u=\{0,1,\ldots,p-1\} \tag{3}$$

$$l\psi_u(x)=\sum_{v=0}^{p+q-1}l\psi_{u,v+1}\,x^v \quad for\ u=\{0,1,\ldots,q-1\} \tag{4}$$

where $X_0,\ldots,X_{q-1}$ are the q collocation points and x_{z+u}, $u=0,1,2,\ldots,p-1$ are the p arbitrarily chosen interpolation points.
To get $\phi_u(x)$ and $\psi_u(x)$, [19] arrived at a matrix equation of the form

$$HE=I \tag{5}$$

where I is the square matrix of dimension $(p+q)\times(p+q)$ while H and E are matrices defined as

$$H=\begin{bmatrix}\phi_{0,1} & \phi_{1,1} & \cdots & \phi_{z-1,1} & l\psi_{0,1} & \cdots & l\psi_{q-1,1}\\ \phi_{0,2} & \phi_{1,2} & \cdots & \phi_{z-1,2} & l\psi_{0,2} & \cdots & l\psi_{q-1,2}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \phi_{0,p+q} & \phi_{1,p+q} & \cdots & \phi_{z-1,p+q} & l\psi_{0,p+q} & \cdots & l\psi_{q-1,p+q}\end{bmatrix} \tag{6}$$

$$E=\begin{bmatrix}1 & X_z & X^2 & \cdots & X_z^{p+q-1}\\ \vdots & X_{z+1} & X_{z+1}^2 & \cdots & X_{z+1}^{p+q-1}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & X_{z+p-1} & X_{z+p-1}^2 & \cdots & X_{z+p-1}^{p+q-1}\\ 0 & 1 & 2X_0 & \cdots & (p+q-1)X_0^{p+q-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 1 & 2X_{q-1} & \cdots & (p+q-1)X_{q-1}^{p+q-2}\end{bmatrix} \tag{7}$$

It follows from (5) that the columns of $H=E^{-1}$ give the continuous coefficients of the continuous scheme (2).
Subsequently, the continuous representations of the BBDFMs for step numbers k = 2, 3 and 4 shall be obtained using multistep collocation method of [19].

2.2. Formulation of Block Backward Differentiation Formulae Methods (BBDFMs) for k = 2

In this section, the interpolation point $p=2$ and the collocation points $q=1$ are considered, therefore (2) becomes

$$y(x)=\phi_0(x)y_z+\phi_1(x)y_{z+1}+l\psi_2(x)f_{z+2} \tag{8}$$

The matrix E in (5) becomes

$$E=\begin{bmatrix}1 & x_z & x_z^2\\ 1 & x_{z+1} & (x_z+l)^2\\ 0 & 1 & 2x_{2+4l}\end{bmatrix} \tag{9}$$

The inverse of the matrix $H=E^{-1}$ is computed using Maple 18 from which the continuous scheme is obtained using (2), evaluating and simplifying it at $x=x_{z+2}$ and its derivative at $=x_{z+1}$, the following discrete schemes are obtained

$$\begin{aligned}y_{z+1}&=\frac{3}{2}lf_{z+1}-\frac{1}{2}lf_{z+2}+y_z\\ y_{z+2}&=-\frac{1}{3}y_z+\frac{4}{3}y_{z+1}+\frac{2}{3}lf_{z+2}\end{aligned} \tag{10}$$

2.3. Formulation of Block Backward Differentiation Formulae Methods (BBDFMs) for k = 3

In this section, the number of interpolation point *p* =3 and the number of collocation points is *q* =1. Therefore, (2) becomes

$$y(x)=\phi_0(x)y_z+\phi_1(x)y_{z+1}+\phi_2(x)y_{z+2}+l\psi_3(x)f_{z+3} \tag{11}$$

The matrix *E* in (5) becomes

$$E=\begin{pmatrix}1 & x_z & x_z^2 & x_z^3\\ 1 & x_{z+1} & (x_z+l)^2 & (x_z+l)^3\\ 1 & x_{z+2l} & (x_z+2l)^2 & (x_z+2l)^3\\ 0 & 1 & 2x_{z+6l} & 3(x_z+3l)^2\end{pmatrix} \tag{12}$$

The inverse of the matrix $H=E^{-1}$is computed using Maple 18 from which the continuous scheme is derived using (2), evaluating and simplifying it at $x=x_{z+3}$ and its derivatives at $x=x_{z+1}$, $x=x_{z+2}$ the following discrete schemes are obtained

$$\begin{aligned}y_{z+1}&=-\frac{11}{4}lf_{z+1}-\frac{1}{4}lf_{z+3}-y_z+2y_{z+2}\\ y_{z+2}&=\frac{22}{23}lf_{z+2}-\frac{4}{23}lf_{z+3}-\frac{5}{23}y_z+\frac{28}{23}y_{z+1}\\ y_{z+3}&=\frac{2}{11}y_z-\frac{9}{11}y_{z+1}+\frac{18}{11}y_{z+2}+\frac{6}{11}lf_{z+3}\end{aligned} \tag{13}$$

2.4. Formulation of Block Backward Differentiation Formulae Methods (BBDFMs) for k =4

Here also, the number of interpolation point, *p* =4 and the number of collocation points, 1 *q* =1.
Therefore, (2) becomes

$$y(x)=\phi_0(x)y_z+\phi_1(x)y_{z+1}+\phi_2(x)y_{z+2}+\phi_3(x)y_{z+3}+l\psi_4(x)f_{z+4} \tag{14}$$

Also the matrix E in (5) becomes

$$E=\begin{pmatrix}1 & x_z & x_z^2 & x_z^3 & x_z^4\\ 1 & x_{z+1} & (x_z+l)^2 & (x_z+l)^3 & (x_z+l)^4\\ 1 & x_{z+2l} & (x_z+2l)^2 & (x_z+2l)^3 & (x_z+2l)^4\\ 0 & x_{z+3l} & (x_z+3l)^2 & (x_z+3l)^3 & (x_z+3l)^4\\ 0 & 1 & 2x_{z+8l} & 3(x_z+4l)^2 & 4(x_z+4l)^3\end{pmatrix} \tag{15}$$

The inverse of the matrix $H=E^{-1}$ is computed using Maple 18 from which the continuous scheme is also derived using (2), evaluating and simplifying it at $x=x_{z+4}$ and its derivatives at $x=x_{z+1}$, $x=x_{z+2}$, $x=x_{z+3}$ the following discrete schemes are obtained

$$\begin{aligned}y_{z+1}&=-\frac{50}{39}lf_{z+1}+\frac{2}{39}lf_{z+4}-\frac{1}{3}y_z+\frac{23}{13}y_{z+2}-\frac{17}{39}y_{z+3}\\ y_{z+2}&=\frac{25}{3}lf_{z+2}+\frac{1}{3}lf_{z+4}-\frac{7}{9}y_z+6y_{z+1}-\frac{38}{9}y_{z+3}\\ y_{z+3}&=\frac{150}{197}lf_{z+3}-\frac{18}{197}lf_{z+4}+\frac{17}{197}y_z-\frac{99}{197}y_{z+1}+\frac{279}{197}y_{z+2}\\ y_{z+4}&=-\frac{3}{25}y_z+\frac{16}{25}y_{z+1}-\frac{36}{25}y_{z+2}+\frac{48}{25}y_{z+3}+\frac{12}{25}lf_{z+4}\end{aligned} \tag{16}$$

3. Analysis of the Basic Properties of the Method

In numerical analysis, it is necessary that the method satisfies the necessary and sufficient conditions as proposed by.[20,21] Therefore, the analysis of order, error constant, consistency, zero stability and region of absolute stability of (10), (13) and (16) are investigated.

3.1. Order and Error Constant

In [20], the Linear Multistep Method is said to be of order d if $C_0 = C_1 = 0, \dots, C_d = 0$ but $C_{d+1} \neq 0$ and C_{d+1} is called the error constant. The order and error constants for (10) are obtained as follows

$$C_0 = C_1 = C_2 = (0 \quad 0)^T$$

$$C_3 = \begin{pmatrix} \frac{5}{12} & -\frac{2}{9} \end{pmatrix}^r$$

Hence, (10) has an order d=2 and error constant $\begin{pmatrix} \frac{5}{12} & -\frac{2}{9} \end{pmatrix}^r$

Using the same techniques, (13) can be presented as follows

$$C_0 = C_1 = C_2 = C_3 = (0 \quad 0 \quad 0)^T$$

$$C_4 = \begin{pmatrix} \frac{7}{24} & \frac{17}{138} & -\frac{3}{22} \end{pmatrix}^T$$

Therefore, (13) has order d=3 and error constant $C_4 = \begin{pmatrix} \frac{7}{24} & \frac{17}{138} & -\frac{3}{22} \end{pmatrix}^T$

Following the same step, (16) can be presented as

$$C_0 = C_1 = C_2 = C_3 = C_4 = (0 \quad 0 \quad 0 \quad 0)^T$$

$$C_5 = \begin{pmatrix} -\frac{29}{390} & -\frac{31}{90} & \frac{111}{1970} & -\frac{12}{125} \end{pmatrix}^T$$

Therefore, (16) has order d=4 and error constants $\begin{pmatrix} -\frac{29}{390} & -\frac{31}{90} & \frac{111}{1970} & -\frac{12}{125} \end{pmatrix}^T$

3.2. Consistency

According to [20], a numerical method is said to be consistent if the order d is greater than 1 i.e. $d \geq 1$. Since the order of our proposed method BBDFM as analyzed for the discrete schemes (10), (13) and (16) in section (3.1) is greater than 1 i.e. $d \geq 1$, the necessary condition for consistency of our proposed method is satisfied. Hence the method is consistent.

3.3. Zero Stability Analysis

In [21], a numerical method is said to be zero stable if the roots r_s, s= 1,2,3,...,n of the first characteristic polynomial $\eta(r)$ expressed as $\eta(r) = det\left(rT_2^{(1)} - T_1^{(1)}\right)$ satisfies $|r_s| \leq 1$ and the roots $|r_s|$ is simple or distinct.

The zero stability for (10) is examined as follows

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_{z+1} \\ y_{z+2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} y_{z-1} \\ y_z \end{pmatrix} + 1\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} f_{z+1} \\ f_{z+2} \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_{z-1} \\ f_z \end{pmatrix}$$

$$T_2^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T_1^{(1)} = \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{3} \end{pmatrix} \text{ and } D_2^{(1)} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{2}{3} \end{pmatrix}$$

The first characteristic polynomial is stated as

$$\eta(r) = det\left(rT_2^{(1)} - T_1^{(1)}\right)$$

$$\left|rT_2^{(1)} - T_1^{(1)}\right| = 0 \qquad (17)$$

Now we have,

$$\eta(r) = \left| r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{3} \end{pmatrix} \right| = \left| \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{3} \end{pmatrix} \right|$$

$$\Rightarrow \eta(r) = \begin{pmatrix} r & -1 \\ 0 & r - \frac{1}{3} \end{pmatrix}$$

Using Maple (18) software, we obtain

$$\eta(r) = \frac{1}{3}r(3r - 1)$$

$$\Rightarrow \frac{1}{3}\mathrm{r}(3\mathrm{r}-1)=0$$

$\Rightarrow \mathrm{r}_1=1, \mathrm{r}_2=\frac{1}{3}$. Since $|\mathrm{r}_s|\le 1$ s= 1, 2, the discrete schemes in (10) is zero stable.

Following the same step for (13), we have

$$\begin{pmatrix} 1 & 2 & 0 \\ -\frac{28}{23} & 1 & 0 \\ \frac{9}{11} & -\frac{18}{11} & 1 \end{pmatrix}\begin{pmatrix} y_{z+1} \\ y_{z+2} \\ y_{z+3} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{5}{23} \\ 0 & 0 & -\frac{2}{11} \end{pmatrix}\begin{pmatrix} y_{z-2} \\ y_{z-1} \\ y_z \end{pmatrix}+1\begin{pmatrix} -\frac{11}{4} & 0 & -\frac{1}{4} \\ 0 & \frac{22}{23} & -\frac{4}{23} \\ 0 & 0 & \frac{6}{11} \end{pmatrix}\begin{pmatrix} f_{z+1} \\ f_{z+2} \\ f_{z+3} \end{pmatrix}+1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{z-2} \\ f_{z-1} \\ f_z \end{pmatrix}$$

Where

$$T_2^{(2)}=\begin{pmatrix} 1 & 2 & 0 \\ -\frac{28}{23} & 1 & 0 \\ \frac{9}{11} & -\frac{18}{11} & 1 \end{pmatrix}, T_1^{(2)}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{5}{23} \\ 0 & 0 & -\frac{2}{11} \end{pmatrix}, D_2^{(2)}=\begin{pmatrix} -\frac{11}{4} & 0 & -\frac{1}{4} \\ 0 & \frac{22}{23} & -\frac{4}{23} \\ 0 & 0 & \frac{6}{11} \end{pmatrix}$$

The first characteristic polynomial is stated as

$$\eta(r)=det\left(rT_2^{(2)}-T_1^{(2)}\right)$$

$$\left|rT_2^{(2)}-T_1^{(2)}\right|=0 \tag{18}$$

Now we have,

$$\eta(r)=\left|r\begin{pmatrix} 1 & 2 & 0 \\ -\frac{28}{23} & 1 & 0 \\ \frac{9}{11} & -\frac{18}{11} & 1 \end{pmatrix}-\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{5}{23} \\ 0 & 0 & -\frac{2}{11} \end{pmatrix}\right|=\left|\begin{pmatrix} r & 2r & 0 \\ -\frac{28}{23}r & r & 0 \\ \frac{9}{11}r & -\frac{18}{11}r & r \end{pmatrix}-\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{5}{23} \\ 0 & 0 & -\frac{2}{11} \end{pmatrix}\right|$$

$$\Rightarrow \eta(\mathrm{r})=\begin{pmatrix} \mathrm{r} & 2\mathrm{r} & -1 \\ -\frac{28}{23}r & \mathrm{r} & -\frac{5}{23} \\ \frac{9}{11}r & -\frac{18}{11}r & r+\frac{2}{11} \end{pmatrix}$$

The following are obtained using Maple (18) software,

$$\eta(\mathrm{r})=\frac{79}{23}\mathrm{r}^3-\frac{29}{23}r^2$$

$$\Rightarrow \frac{79}{23}\mathrm{r}^3-\frac{29}{23}r^2=0$$

$\Rightarrow \mathrm{r}_1=\frac{29}{79}, \mathrm{r}_2=0, \mathrm{r}_3=0$. Since $|\mathrm{r}_s|\le 1$ s= 1, 2, 3, the discrete schemes in (13) is zero stable.

By the same technique (16) can be presented as follows

$$\begin{pmatrix} 1 & -\frac{23}{13} & \frac{17}{39} & 0 \\ -6 & 1 & \frac{38}{9} & 0 \\ \frac{99}{197} & -\frac{279}{197} & 1 & 0 \\ -\frac{16}{25} & \frac{36}{25} & -\frac{48}{25} & 1 \end{pmatrix}\begin{pmatrix} y_{z+1} \\ y_{z+2} \\ y_{z+3} \\ y_{z+4} \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{7}{9} \\ 0 & 0 & 0 & -\frac{17}{197} \\ 0 & 0 & 0 & \frac{3}{25} \end{pmatrix}\begin{pmatrix} y_{z-3} \\ y_{z-2} \\ y_{z-1} \\ y_z \end{pmatrix}+1\begin{pmatrix} -\frac{50}{39} & 0 & 0 & \frac{2}{39} \\ 0 & \frac{25}{23} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{150}{197} & -\frac{18}{197} \\ 0 & 0 & 0 & \frac{12}{25} \end{pmatrix}\begin{pmatrix} f_{z+1} \\ f_{z+2} \\ f_{z+3} \\ f_{z+4} \end{pmatrix}+1\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{z-3} \\ f_{z-2} \\ f_{z-1} \\ f_z \end{pmatrix}$$

where

$$T_2^{(3)}=\begin{pmatrix} 1 & -\frac{23}{13} & \frac{17}{39} & 0 \\ -6 & 1 & \frac{38}{9} & 0 \\ \frac{99}{197} & -\frac{279}{197} & 1 & 0 \\ -\frac{16}{25} & \frac{36}{25} & -\frac{48}{25} & 1 \end{pmatrix}, T_1^{(3)}=\begin{pmatrix} 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{7}{9} \\ 0 & 0 & 0 & -\frac{17}{197} \\ 0 & 0 & 0 & \frac{3}{25} \end{pmatrix}, D_2^{(3)}=\begin{pmatrix} -\frac{50}{39} & 0 & 0 & \frac{2}{39} \\ 0 & \frac{25}{23} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{150}{197} & -\frac{18}{197} \\ 0 & 0 & 0 & \frac{12}{25} \end{pmatrix}$$

In this case, the first characteristic polynomial is stated as

$$\eta(r)=det\left(rT_2^{(3)}-T_1^{(3)}\right)$$

$$\left|rT_2^{(3)}-T_1^{(3)}\right|=0 \tag{19}$$

Now we have,

$$\eta(r)=\left|r\begin{pmatrix}1 & -\frac{23}{13} & \frac{17}{39} & 0\\ -6 & 1 & \frac{38}{9} & 0\\ \frac{99}{197} & -\frac{279}{197} & 1 & 0\\ -\frac{16}{25} & \frac{36}{25} & -\frac{48}{25} & 1\end{pmatrix}-\begin{pmatrix}0&0&0&\frac{1}{3}\\0&0&0&\frac{7}{9}\\0&0&0&-\frac{17}{197}\\0&0&0&\frac{3}{25}\end{pmatrix}\right|=\left|\begin{pmatrix}r & -\frac{23}{13}r & \frac{17}{39}r & 0\\ -6r & r & \frac{38}{9}r & 0\\ \frac{99}{197}r & -\frac{279}{197}r & r & 0\\ -\frac{16}{25}r & \frac{36}{25}r & -\frac{48}{25}r & r\end{pmatrix}-\begin{pmatrix}0&0&0&\frac{1}{3}\\0&0&0&\frac{7}{9}\\0&0&0&-\frac{17}{197}\\0&0&0&\frac{3}{25}\end{pmatrix}\right|$$

$$\Rightarrow \eta(r)=\begin{pmatrix}r & -\frac{23}{13}r & \frac{17}{39}r & -\frac{1}{3}\\ -6r & r & \frac{38}{9}r & -\frac{7}{9}\\ \frac{99}{197}r & -\frac{279}{197}r & r & \frac{17}{197}\\ -\frac{16}{25}r & \frac{36}{25}r & -\frac{48}{25}r & r-\frac{3}{25}\end{pmatrix}$$

The following are obtained using Maple (18) software,

$$\eta(r)=-\frac{10000}{2561}r^4-\frac{10000}{2561}r^3$$

$$\Rightarrow -\frac{10000}{2561}r^4-\frac{10000}{2561}r^3=0$$

$\Rightarrow r_1=-1, r_2=0, r_3=0, r_4=0$. Since $|r_s|\leq 1$ s= 1, 2, 3,4, the discrete schemes in (16) is zero stable.

3.4. Convergence

Theorem 2: The necessary and sufficient condition for a linear multistep method to be convergent is that it must be consistent and zero stable as stated by [21]. Since the discrete schemes (10), (13) and (16) are both consistent and zero stable, therefore the method is convergent.

3.5. Region of Absolute Stability

The regions of absolute stability of the numerical methods for SDDEs are considered. We considered finding the M - and R -stability by applying (10), (13) and (16) to the DDEs of this form

$$dX(t)=\alpha(X(t),X(t-\tau),t)dt+\beta(X(t),X(t-\tau),t)dW(t) \text{ for } t>t_0, \tau>0$$
$$X(t)=\xi(t), \text{ for } t\leq t_0 \tag{20}$$

where $\xi(t)$ is the initial function, α,β are complex coefficients, $\tau=wl$, $w\in Z^+$, l is the step size and $w=\frac{\tau}{l}$, w is a positive integer. Let $A_1=l\alpha$ and $A_2=l\beta$, then from (10) let

$y_{z+2}=\begin{pmatrix}y_{z+1}\\y_{z+2}\end{pmatrix}$, $y_z=\begin{pmatrix}y_{z-1}\\y_z\end{pmatrix}$, $F_{z+2}=\begin{pmatrix}f_{z+1}\\f_{z+2}\end{pmatrix}$ and $F_z=\begin{pmatrix}f_{z-1}\\f_z\end{pmatrix}$

Since

$T_2^{(1)}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $T_1^{(1)}=\begin{pmatrix}0&-1\\0&\frac{1}{3}\end{pmatrix}$ and $D_2^{(1)}=\begin{pmatrix}\frac{3}{2}&-\frac{1}{2}\\0&\frac{2}{3}\end{pmatrix}$

we have,

$$T_2^{(1)}Y_{z+2}=T_1^{(1)}Y_{z+1}-1\sum_{i=1}^{2}D_i^{(1)}F_{z+i} \tag{21}$$

Applying the same technique for (13), we have

$y_{z+3}=\begin{pmatrix}y_{z+1}\\y_{z+2}\\y_{z+3}\end{pmatrix}$, $y_z=\begin{pmatrix}y_{z-2}\\y_{z-1}\\y_z\end{pmatrix}$, $F_{z+3}=\begin{pmatrix}f_{z+1}\\f_{z+2}\\f_{z+3}\end{pmatrix}$ and $F_z=\begin{pmatrix}f_{z-2}\\f_{z-1}\\f_z\end{pmatrix}$

Since

$T_2^{(2)}=\begin{pmatrix}1&2&0\\-\frac{28}{23}&1&0\\\frac{9}{11}&-\frac{18}{11}&1\end{pmatrix}$, $T_1^{(1)}=\begin{pmatrix}0&0&1\\0&0&\frac{5}{23}\\0&0&-\frac{2}{11}\end{pmatrix}$ and $D_2^{(2)}=\begin{pmatrix}-\frac{11}{4}&0&-\frac{1}{4}\\0&\frac{22}{23}&-\frac{4}{23}\\0&0&\frac{6}{11}\end{pmatrix}$

we have,

$$T_2^{(2)}Y_{z+2}=T_1^{(2)}Y_{z+1}-1\sum_{i=1}^{2}D_i^{(2)}F_{z+i} \tag{22}$$

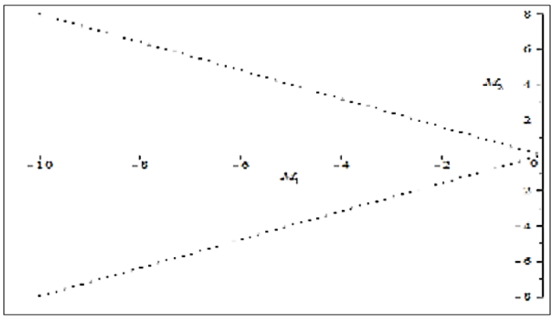

Fig. 1. The *M* - stability region of the schemes in (10)

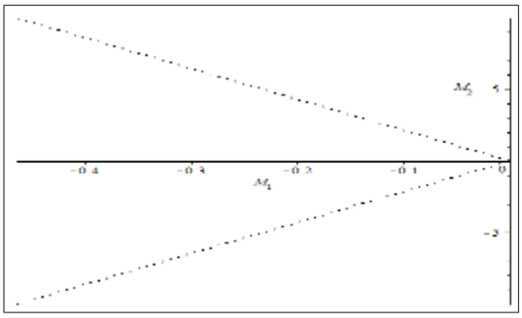

Fig. 2. The *M* - stability region of the schemes in (13)

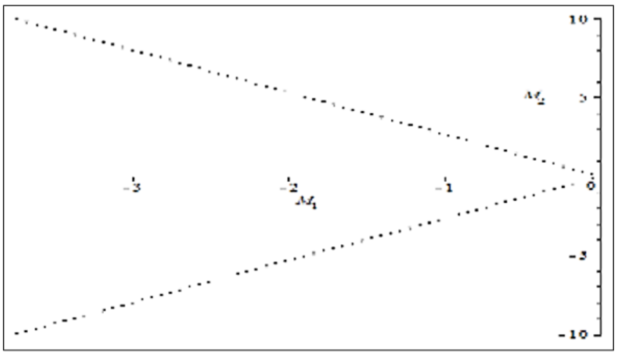

Fig. 3. The *M* - stability region of the schemes in (16)

Applying the same approach for (16), we have

$$y_{z+4}=\begin{pmatrix}y_{z+1}\\y_{z+2}\\y_{z+3}\\y_{z+4}\end{pmatrix},\ y_z=\begin{pmatrix}y_{z-3}\\y_{z-2}\\y_{z-1}\\y_z\end{pmatrix},\ F_{z+4}=\begin{pmatrix}f_{z+1}\\f_{z+2}\\f_{z+3}\\f_{z+4}\end{pmatrix}\text{ and } F_z=\begin{pmatrix}f_{z-3}\\f_{z-2}\\f_{z-1}\\f_z\end{pmatrix}$$

Since

$$T_2^{(3)}\begin{pmatrix}1 & -\frac{23}{13} & \frac{17}{39} & 0\\ -6 & 1 & \frac{38}{9} & 0\\ \frac{99}{197} & -\frac{279}{197} & 1 & 0\\ -\frac{16}{25} & \frac{36}{25} & -\frac{48}{25} & 1\end{pmatrix},\ T_2^{(3)}=\begin{pmatrix}0&0&0&\frac{1}{3}\\0&0&0&\frac{7}{9}\\0&0&0&-\frac{17}{197}\\0&0&0&\frac{3}{25}\end{pmatrix}\text{ and } D_2^{(3)}=\begin{pmatrix}-\frac{50}{39}&0&0&\frac{2}{39}\\0&\frac{25}{23}&0&\frac{1}{3}\\0&0&\frac{150}{197}&-\frac{18}{197}\\0&0&0&\frac{12}{25}\end{pmatrix}$$

we have,

$$T_2^{(3)}Y_{z+2}=T_1^{(3)}Y_{z+1}-1\sum_{i=1}^{2}D_i^{(3)}F_{z+i} \tag{23}$$

From [22], the polynomials of M- and R- stability are constructed by applying (21), (22) and (23) to (20) and (10), (13) and (16) to (20) as presented below

$$\lambda^{(1)}(\mu)=det\left[\left(T_2^{(1)}-A_1D_2^{(1)}\right)\mu^{2+w}-\left(T_1^{(1)}-A_1D_1^{(1)}\right)\mu^{1+w}-A_2\sum_{i=1}^{2}D_i^{(1)}\mu^{v}\right]$$

$$\lambda^{(2)}(\mu)=det\left[\left(T_2^{(2)}-A_1D_2^{(2)}\right)\mu^{2+w}-\left(T_1^{(2)}-A_1D_1^{(2)}\right)\mu^{1+w}-A_2\sum_{i=1}^{2}D_i^{(2)}\mu^{v}\right]$$

$$\lambda^{(3)}(\mu)=det\left[\left(T_2^{(3)}-A_1D_2^{(3)}\right)\mu^{2+w}-\left(T_1^{(3)}-A_1D_1^{(3)}\right)\mu^{1+w}-A_2\sum_{i=1}^{2}D_i^{(3)}\mu^{v}\right]\ and$$

$$\rho^{(1)}(\mu)=det\left[T_2^{(1)}\mu^{2+w}-T_1^{(1)}\mu^{1+w}-A_2\sum_{i=1}^{2}D_i^{(1)}\mu^{v}\right]$$

$$\rho^{(2)}(\mu)=det\left[T_2^{(2)}\mu^{2+w}-T_1^{(2)}\mu^{1+w}-A_2\sum_{i=1}^{2}D_i^{(2)}\mu^{v}\right]$$

$$\rho^{(3)}(\mu)=det\left[T_2^{(3)}\mu^{2+w}-T_1^{(3)}\mu^{1+w}-A_2\sum_{i=1}^{2}D_i^{(3)}\mu^{v}\right]$$

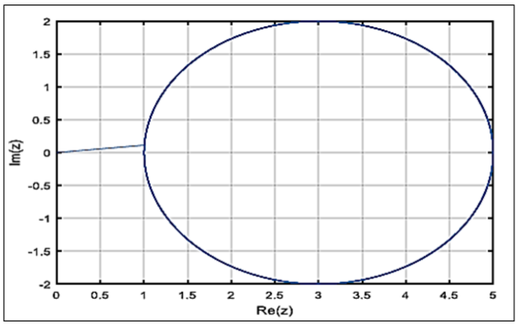

Fig. 4. The *R* - stability region of the schemes in (10)

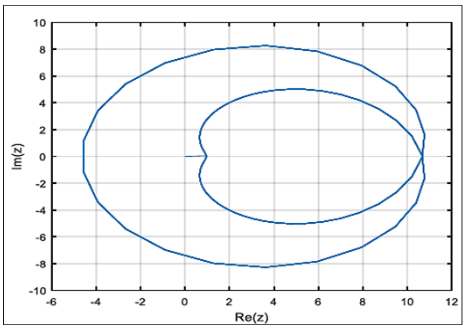

Fig. 5. The *R* - stability region of the schemes in (13)

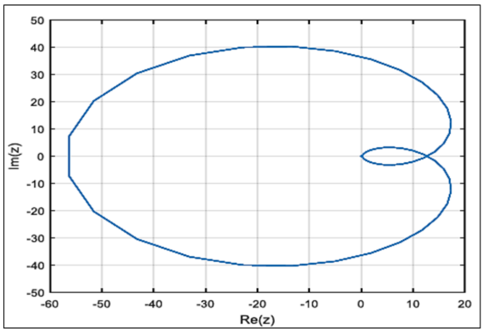

Fig. 6. The *R* - stability region of the schemes in (16)

then the M- and R- stability of (10), (13) and(16) for $w = 1$ are investigated, plotted and presented in figure 1 to 6 using of Maple 18 and MATLAB. The M -stability regions in Figs 1 to 3 lie inside the open-ended region while the R –stability regions in Figs 4 to 6 lie inside the enclosed region.

4. Formulation and Implementation of the Two New Mathematical Expressions for the Evaluations of the Delay Terms

Here, we shall formulate two accurate and efficient mathematical expressions for the evaluation of the delay terms on the drift and the noise term on the diffusion part of the stochastic delay differential equations. The delay term $(t-\tau)$ shall be evaluated with the accurate and efficient formula of this form

$$\xi_{n+j}(t) = \frac{n}{c}\big((cq + (n+j-r-1)h)\big), c \neq 0 \tag{24}$$

Also, we formulated an expression using normalized Brownian Motion to evaluate the noise term $dW(t)$ such that the distribution are Gaussian with $N(0,1)$ whose mean μ is 0 and the standard deviation σ is 1. The random process is expressed as

$$W(t) = \frac{1}{\sqrt{((n+j-r-1)h)\pi}} e^{-t^2 / (n+j-r-1)^h}, for\ t \geq 0 \tag{25}$$

Then by differentiating (22), it gives

$$dW(t) = \frac{-2t}{(n+j-r-1)h\sqrt{((n+j-r-1)h)\pi}} e^{-t^2 / (n+j-r-1)^h}, for\ t \geq 0 \tag{26}$$

where $j \in (-k, k)$, k is a step number, $r = \frac{\tau}{l} \in Z$, $\tau = rl$, τ is the delay term, $z = 0,1,2, \dots, Z-1$ and Z is the number of solutions in the giving interval which is implemented to approximate the delay term (τ) at the point $t = t_z - \tau$ using the previous values of $\xi_{z+j} = \varphi(t)$ at $t_z - \tau \leq 0$ whenever $t_z - \tau > t_0$ where $\xi_{z+j}(t)$ is the approximation to $y(t_z - \tau)$. The results of the above expressions in (21) and (23) shall be obtained using Maple 18 with $z = 0,1,2, \dots, Z-1$ which shall be incorporated into the SDDEs before its evaluation at constant step size l to obtain the numerical solutions of y_z.

Table 1. Solution of Problem 1 using the BBDFM for Step Numbers k = 2, 3 & 4

t	Exact Solution	Numerical Solution K = 2	Numerical Solution K = 3	Numerical Solution K = 4
0.1	0.904837418	0.989721979	0.989795572	0.989834933
0.2	0.818730753	0.975408765	0.97530279	0.975272659
0.3	0.740818221	0.9516706	0.96061934	0.960690853
0.4	0.670320046	0.936658324	0.919486611	0.946484521
0.5	0.60653066	0.88037731	0.904403141	0.879622773
0.6	0.548811636	0.863394864	0.891872587	0.86457778
0.7	0.496585304	0.782513608	0.78247501	0.852397377
0.8	0.449328964	0.764636565	0.765831259	0.838628718
0.9	0.40656966	0.668308945	0.757776391	0.668360541
1	0.367879441	0.650583746	0.608806279	0.652582126
1.1	0.332871084	0.548435037	0.592588258	0.644933964
1.2	0.301194212	0.531806288	0.588560073	0.632108604
1.3	0.272531793	0.432450784	0.433079062	0.43310202
1.4	0.246596964	0.417642641	0.419008082	0.419651106
1.5	0.22313016	0.32765197	0.417878601	0.415887702
1.6	0.201896518	0.31510572	0.281676027	0.405822446
1.7	0.182683524	0.238536316	0.27072879	0.239390617
1.8	0.165298888	0.228405613	0.271224107	0.229902429
1.9	0.149568619	0.166864098	0.167511394	0.228581313
2	0.135335283	0.159057885	0.159838526	0.221901369
2.1	0.122456428	0.112160267	0.160929513	0.112881121
2.2	0.110803158	0.106414411	0.091089721	0.107295817
2.3	0.100258844	0.072440919	0.08622929	0.107083658
2.4	0.090717953	0.068397676	0.087293898	0.10332957
2.5	0.082084999	0.044957145	0.045294506	0.045414953
2.6	0.074273578	0.042235407	0.04250546	0.042656512
2.7	0.067205513	0.026809236	0.043289802	0.042760171
2.8	0.060810063	0.025055625	0.020596592	0.040971772
2.9	0.05502322	0.015361802	0.019144331	0.015592349
3	0.049787068	0.014279918	0.019627045	0.014445331

CPU time of BBDFM for k = 2 is 0.10s, k = 3 is 0.05s and k = 4 is 0.003s

Table 2. Solution of Problem 1 using the BBDFM for Step Numbers k = 2, 3 & 4

t	Exact Solution	Numerical Solution K = 2	Numerical Solution K = 3	Numerical Solution K = 4
0.1	1.740818221	1.965585557	1.965738898	1.965821629
0.2	1.548811636	1.92416048	1.92393967	1.92387634
0.3	1.40656966	1.858799456	1.883140294	1.883290605
0.4	1.301194212	1.81889159	1.775134208	1.844819954
0.5	1.22313016	1.681207706	1.737585379	1.679377214
0.6	1.165298888	1.641972392	1.707787389	1.644900542
0.7	1.122456428	1.480188613	1.480405217	1.618426822
0.8	1.090717953	1.44684004	1.44916074	1.588725806
0.9	1.067205513	1.30130178	1.436567278	1.302203337
1	1.049787068	1.276536298	1.229779891	1.279401231
1.1	1.036883167	1.168875114	1.210017712	1.270770391
1.2	1.027323722	1.152757329	1.207639581	1.253602103
1.3	1.020241911	1.085298279	1.086313602	1.086664816
1.4	1.014995577	1.076084293	1.076817528	1.077245903
1.5	1.011108997	1.039739049	1.077623435	1.076116241
1.6	1.008229747	1.035104685	1.027304664	1.069703526
1.7	1.006096747	1.018120617	1.023828629	1.018722861
1.8	1.004516581	1.016067265	1.024647808	1.016305042
1.9	1.003345965	1.009146145	1.009356305	1.016422531
2	1.002478752	1.008343981	1.008384576	1.014890003
2.1	1.001836305	1.0058756	1.00872226	1.006006286
2.2	1.001360368	1.005599103	1.005253738	1.005623912
2.3	1.001007785	1.004826728	1.005045763	1.005683661
2.4	1.000746586	1.004742595	1.005135404	1.005452772
2.5	1.000553084	1.004530177	1.004541355	1.004545624
2.6	1.000409735	1.004507568	1.004507183	1.004508728
2.7	1.000303539	1.004456157	1.004524052	1.004516996
2.8	1.000224867	1.00445079	1.004446569	1.004495414
2.9	1.000166586	1.004439829	1.004442244	1.004440854
3	1.00012341	1.004438703	1.004444583	1.004438708

CPU time of BBDFM for k = 2 is 0.12s, k = 3 is 0.06s and k = 4 is 0.002s

5. Numerical Computations

In this section, the delay term and the noise term shall be evaluated using the two expressions formulated in section four above which shall be incorporated into some stochastic delay differential equations before its evaluation with the discrete schemes (10), (13) and (16) at constant step size to obtain its numerical solutions of y_z.

5.1. Numerical Problems

Problem 1

$$dX(t) = -1000\left(X(t) + X\left(t - \left(In(1000-1)\right)\right)\right)dt + \left(X(t) + X\left(t - \left(In(1000-1)\right)\right)\right)dW(t), 0 \le t \le 3$$

$$\xi(t) = e^{-t}, \quad t \le 0$$

Exact Solution is $\xi(t) = e^{-t}$

Problem 2

$$dX(t) = -1000\big(X(t) + 997e^{-3}X(t-1) + (1000 - 997e^{-3})\big)dt + \big(X(t) + 997e^{-3}X(t-1) + (1000 - 997e^{-3})\big)dW(t), 0 \le t \le 3$$

$$\xi(t) = 1 + e^{-3t}, \quad t \le 0$$

Exact Solution is $\xi(t) = 1 + e^{-3t}$

The above problems were solved using the discrete schemes (10), (13) and (16) as generated by the Block Backward Differentiation Formulae Methods (BBDFMs) and the results obtained are presented in tables 1 to 2.

Table 3. Absolute Errors of BBDFM for k = 2, 3 and 4 using Problem 1

t	k = 2 Error	k = 3 Error	k = 4 Error
0.1	0.084884561	0.084958154	0.084997515
0.2	0.156678012	0.156572037	0.156541906
0.3	0.21085238	0.219801119	0.219872632
0.4	0.266338278	0.249166565	0.276164475
0.5	0.273846651	0.297872481	0.273092113
0.6	0.314583227	0.343060951	0.315766144
0.7	0.285928304	0.285889706	0.355812073
0.8	0.315307601	0.316502295	0.389299754
0.9	0.261739285	0.351206732	0.261790881
1	0.282704304	0.240926838	0.284702685
1.1	0.215563953	0.259717174	0.312062881
1.2	0.230612076	0.287365861	0.330914392
1.3	0.159918991	0.160547269	0.160570227
1.4	0.171045677	0.172411118	0.173054142
1.5	0.104521809	0.194748441	0.192757542
1.6	0.113209202	0.079779509	0.203925928
1.7	0.055852792	0.088045266	0.056707093
1.8	0.063106725	0.105925219	0.064603541
1.9	0.017295479	0.017942775	0.079012694
2	0.023722601	0.024503243	0.086566085
2.1	0.010296162	0.038473085	0.009575307
2.2	0.004388747	0.019713437	0.003507341
2.3	0.027817925	0.014029554	0.006824814
2.4	0.022320278	0.003424056	0.012611617
2.5	0.037127854	0.036790493	0.036670046
2.6	0.032038171	0.031768118	0.031617066
2.7	0.040396277	0.023915711	0.024445342
2.8	0.035754438	0.04021347	0.019838291
2.9	0.039661418	0.035878889	0.039430871
3	0.03550715	0.030160023	0.035341738

Table 4. Absolute Errors of BBDFM for k = 2, 3 and 4 using Problem2

t	k = 2 Error	k = 3 Error	k = 4 Error
0.1	0.224767336	0.224920677	0.225003408
0.2	0.375348844	0.375128034	0.375064704
0.3	0.452229796	0.476570634	0.476720945
0.4	0.517697378	0.473939996	0.543625742
0.5	0.458077546	0.514455219	0.456247054
0.6	0.476673504	0.542488501	0.479601654
0.7	0.357732185	0.357948789	0.495970394
0.8	0.356122087	0.358442787	0.498007853
0.9	0.234096267	0.369361765	0.234997824
1	0.22674923	0.179992823	0.229614163
1.1	0.131991947	0.173134545	0.233887224
1.2	0.125433607	0.180315859	0.226278381
1.3	0.065056368	0.066071691	0.066422905
1.4	0.061088716	0.061821951	0.062250326
1.5	0.028630052	0.066514438	0.065007244
1.6	0.026874938	0.019074917	0.061473779
1.7	0.01202387	0.017731882	0.012626114
1.8	0.011550684	0.020131227	0.011788461
1.9	0.00580018	0.00601034	0.013076566
2	0.005865229	0.005905824	0.012411251
2.1	0.004039295	0.006885955	0.004169981
2.2	0.004238735	0.00389337	0.004263544
2.3	0.003818943	0.004037978	0.004675876
2.4	0.003996009	0.004388818	0.004706186
2.5	0.003977093	0.003988271	0.00399254
2.6	0.004097833	0.004097448	0.004098993
2.7	0.004152618	0.004220513	0.004213457
2.8	0.004225923	0.004221702	0.004270547
2.9	0.004273243	0.004275658	0.004274268
3	0.004315293	0.004321173	0.004315298

6. Results and Discussions

Here, the numerical solutions obtained after solving some SDDEs using the schemes derived in (10), (13) and (16) shall be analyzed by computing their absolute errors.

6.1. Analysis of Results

The analysis of results is obtained by determining absolute differences of the exact solutions and the numerical solutions. The results are presented in the tables 3 to 4.

6.2. Comparison of Results

In order to determine the accuracy, efficiency and advantage of our method BBDFM over other existing methods, we shall compare the methods in two ways using the following notations

BBDFM = Block Backward Differentiation Formulae Methods (BBDFMs) for step numbers k = 2, 3 and 4.

CSSEMM = Continuous Split-Step Scheme Euler-Maruyama Method for step numbers k = 2, 3 and 4 in [1].

EMM = Euler-Maruyama Method for step numbers k = 2, 3 and 4 in [3].

MAXE = Maximum Error. The maximum error (MAXE) is the highest value of the absolute error for total number of steps taken and the numerical method with the highest Maximum Error is the method that performs better.

6.2.1. Comparison of our Method BBDFM with other Numerical Methods Applied in Solving Stochastic Delay Differential Equations using Interpolation Techniques in Evaluating the Delay Terms (Table 5, 6)

Table 5. Comparison between the Maximum Errors of our method BBDFM k = 2, 3 and 4 with [1, 3] for constant step size l = 0.01 for Problem 1

Numerical Method	COMPARED MAXEs with [1,3]
BBDFM MAXE for k = 2	3.15E-01
BBDFM MAXE for k = 3	3.51E-01
BBDFM MAXE for k = 4	3.89E-01
CSSEMM MAXE for k = 2	4.76E-02
CSSEMM MAXE for k = 3	9.17E-02
CSSEMM MAXE for k = 4	1.62E-01
EMM MAXE for k = 2	1.84E-02
EMM MAXE for k = 3	4.04E-03
EMM MAXE for k = 4	9.73E-04

Table 6. Comparison between the Maximum Errors of our method BBDFM k =2, 3 and 4 with [1, 3] for constant step size l = 0.01 for Problem 2

Numerical Method	COMPARED MAXEs with [1,3]
BBDFM MAXE for k = 2	5.18E-01
BBDFM MAXE for k = 3	5.42E-01
BBDFM MAXE for k = 4	5.44E-01
CSSEMM MAXE for k = 2	3.18E-02
CSSEMM MAXE for k = 3	5.90E-02
CSSEMM MAXE for k = 4	5.90E-02
EMM MAXE for k = 2	1.09E-01
EMM MAXE for k = 3	4.91E-02
EMM MAXE for k = 4	2.44E-02

6.2.2. Comparison of our Method with other Numerical Methods Applied in Solving First Order Delay Differential Equations Using the Formula Developed by [16] in Evaluating the Delay Term (Tables 7, 8)

In this section, the notations used are stated as follows

ESDBBDFM = Extended Second Derivative Block Backward Differentiation Formulae Method for step numbers k = 2, 3 and 4 in [11]

RBBDFM = Reformulated Block Backward Differentiation Formulae Methods for step numbers k = 3 and 4 in [16].

Table 7. Comparison between the Maximum Errors of our Method BBDFM k = 2, 3 and 4 and [11, 16] for constant step size l = 0.01 for Problem 1

Numerical Method	COMPARED MAXEs with [11,16]
BBDFM MAXE for k = 2	3.15E-01
BBDFM MAXE for k = 3	3.51E-01
BBDFM MAXE for k = 4	3.89E-01
ESDBBDFM MAXE for k = 2	5.04E-02
ESDBBDFM MAXE for k = 3	6.69E-02
ESDBBDFM MAXE for k = 4	7.09E-02
RBBDFM MAXE for k = 3	1.54E-09
RBBDFM MAXE for k = 4	1.04E-09

Table 8. Comparison between the Maximum Errors of our Method BBDFM k = 2, 3 and 4 and [11, 16] for constant step size l = 0.01 for Problem 2

Numerical Method	COMPARED MAXEs with [11,16]
BBDFM MAXE for k = 2	5.18E-01
BBDFM MAXE for k = 3	5.42E-01
BBDFM MAXE for k = 4	5.44E-01
ESDBBDFM MAXE for k = 2	2.08E-03
ESDBBDFM MAXE for k = 3	1.91E-03
ESDBBDFM MAXE for k = 4	3.43E-03
RBBDFM MAXE for k = 3	4.88E-06
RBBDFM MAXE for k = 4	4.38E-06

7. Conclusions

In this study, we have shown that Backward Differentiation Formulae Methods (BBDFMs) can be used to solve Stochastic Delay Differential Equations (SDDEs) using the two new expressions as formulated in section four above (24) and (26) for evaluations of the delay term and the noise term. To prove the advantage of the method BBDFM, we compare the performances of our method with other existing methods which revealed that BBDFM performed better in terms of efficiency, accuracy, consistency, convergence and region of absolute stability at constant step size l as showed in table 5 to 6. Also, it was observed in tables 1 to 2 that the discrete schemes of higher step number k = 4 of BBDFM performed slightly better and faster than the lower step numbers of k = 2 and 3 when compared with the exact solutions. Hence, it is recommended that BBDFM schemes for step numbers k = 2, 3, and 4 are suitable for solving SDDEs. Further research should be carried-out for step numbers k = 5, 6, 7… on the construction of discrete schemes of BBDFM for numerical solutions of SDDEs.

Conflicts of Interest

The authors declare no conflict of interest.

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