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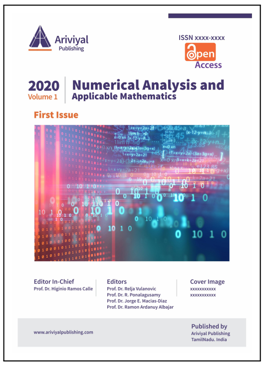

Maximum Norm a Posteriori Error Estimation for a System of Singularly Perturbed Semilinear Reaction-Diffusion Equations

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Abstract: In this paper, a system of singularly perturbed semilinear reaction-diffusion equations is discretized on an arbitrary nonuniform mesh using an upwind finite difference scheme. Based on the quadratic polynomial interpolation function, a second-order maximum norm a posteriori error estimation is derived. Then, this bound is used to design an adaptive grid generation algorithm. Numerical results are given to verify the effectiveness of the presented adaptive grid method.

Keywords: singularly perturbed; reaction-diffusion; adaptive grid; a posteriori error estimation

1. Introduction

This paper is devoted to the problem of finding a numerical approximation to the solution $\mathbf{u} = (u_1, \cdots, u_m)^T \in (C^2(0,1) \cap C[0,1])^m$ of the following system of singularly perturbed semilinear reaction-diffusion equations

$$\mathbf{Lu} \equiv -\varepsilon \mathbf{u}'' + \mathbf{a}(x, \mathbf{u}) = \mathbf{f}, \quad x \in \Omega = (0,1), \tag{1}$$

$$\mathbf{u}(0) = \mathbf{p}, \mathbf{u}(1) = \mathbf{q}, \tag{2}$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter and $\mathbf{a}(x, \mathbf{u}) = (a_1(x, \mathbf{u}), \cdots, a_m(x, \mathbf{u}))^T$, $\mathbf{f} = (f_1, \cdots, f_m)^T$ are given function vectors. The source terms $f_j \in C(\bar{\Omega})$, $j = 1, \cdots, m$. For all $(x, \mathbf{u}) \in \bar{\Omega} \times \mathbb{R}^m$ and a given constant α, we assume that the nonlinear term $\mathbf{a}(x, \mathbf{u})$ satisfies

$$\frac{\partial a_i}{\partial u_i} > \sum_{j \neq i, j=1}^{m} \left| \frac{\partial a_i}{\partial u_j} \right| > 0, \text{ for } 1 \le i \le m, \tag{3}$$

$$\frac{\partial a_i}{\partial u_j} \le 0, \text{ for } i \neq j, \ \sum_{j=1}^{m} \frac{\partial a_i}{\partial u_j} > \alpha > 0, \ i = 1, \cdots, m. \tag{4}$$

It is well known that system of singularly perturbed problems are widespread in optimal control problem[1] and in the modeling of resistance-capacitor electrical circuits. Over the last decades, some special numerical approaches for linear system of singularly perturbed reaction-diffusion equations have been discussed in the literature (see, e.g.,[3, 4, 5, 6, 7, 8, 9, 10]). Meanwhile, many researchers[11, 12, 13] developed some layer-adapted mesh methods for the system of semilinear singularly perturbed reaction-diffusion equations.

As far as we know, adaptive grid methods based on an a posteriori error estimation have already been applied to scalar singularly perturbed problems by several researchers (e.g.,[14, 15, 16]). Meanwhile, these approaches were also widely used in solving the system of singularly perturbed differential equations[17, 18, 19] The numerical method based on the adapted a posteriori mesh for a system of singularly perturbed reaction-diffusion equations are not found in the literature to the best of our knowledge.

This paper is concerned with the numerical scheme based on a reliable a posteriori mesh for the problem (1)-(2). An upwind finite difference scheme is constructed on an arbitrary nonuniform mesh. A posteriori error analysis in the maximum norm is derived based on the polynomial interpolation technique and the stability result for the exact solution $\mathbf{u}(x)$. Based on this a posteriori error estimation and mesh equidistribution principle, a suitable monitor function is chosen to design an adaptive grid algorithm. Numerical results are provided to support our theoretical estimation. *Notations.* Throughout this paper, let C be a generic positive constant that is independent of perturbation parameter ε and mesh parameter N. It may take different values in different place.

In our estimates, we use the L_∞ norm defined by $\|v\|_{\infty,\bar{\Omega}} = \operatorname{ess\,sup}_{x\in\bar{\Omega}} |v(x)|$, where ess sup denotes the essential supremum. For vector-valued function $\mathbf{v} = (v_1(x), \cdots, v_m(x))^T$,we define $\|\mathbf{v}\|_{\infty,\bar{\Omega}} = \max\{\|v_1\|_{\infty,\bar{\Omega}}, \cdots, \|v_m\|_{\infty,\bar{\Omega}}\}$. For a real-valued mesh function $\varphi := \{\varphi(x_i)\}_{i=0}^N$, define the discrete maximum norm $\|\varphi\|_\infty = \max_{i=0,1,\cdots,N} |\varphi(x_i)|$. For vector mesh functions $\mathbf{V} := \{(V_1(x_i), \cdots, V_m(x_i))^T\}_{i=0}^N$, we set $\|\mathbf{V}\|_\infty = \max\{\|V_1\|_\infty, \cdots, \|V_m\|_\infty\}$.

2. Preliminary results

Since the differential operator $\mathbf{L}$ is a nonlinear operator, we first give the linearized form of the problem (1) as follows:

$$\mathbf{L}\mathbf{u} \equiv -\varepsilon \mathbf{u}'' + \mathbf{B}(x)\mathbf{u}(x) = -\mathbf{a}(x, \mathbf{0}) + \mathbf{f}, \quad x \in \Omega = (0, 1), \tag{5}$$

where $\mathbf{B}(x) = \frac{\partial \mathbf{a}(x,\gamma\mathbf{u})}{\partial \mathbf{u}}$, $0 < \gamma < 1$. Since the differential operator $\mathcal{L}$ satisfies a maximum principle (see Theorem 1 of [2]) the problem (1)-(2) has a unique solution $\mathbf{u} = (u_1, \cdots, u_m)^T$. Furthermore, we have the following stability result for the exact solution $\mathbf{u}$.

Lemma 2.1. *Let $\boldsymbol{u}(x)$ be the solution of the problem (1)-(2). Then*

$$\|\boldsymbol{u}\|_{\infty,\bar{\Omega}} \le \max\{\|\boldsymbol{u}(0)\|_\infty, \|\boldsymbol{u}(1)\|_\infty\} + \frac{1}{\alpha}\left(\|\boldsymbol{a}(\cdot, 0)\|_{\infty,\bar{\Omega}} + \|\boldsymbol{f}\|_{\infty,\bar{\Omega}}\right). \tag{6}$$

Proof. The proof is given in [2, Corollary 2]. □

Corollary 2.1.1. *For any two function vectors $\boldsymbol{v}(x)$ and $\boldsymbol{w}(x)$ satisfying*

$$\boldsymbol{v}(0) = \boldsymbol{w}(0), \quad \boldsymbol{v}(1) = \boldsymbol{w}(1)$$

and

$$\boldsymbol{L}\boldsymbol{v}(x) - \boldsymbol{L}\boldsymbol{w}(x) = \boldsymbol{F}(x),$$

where $\boldsymbol{F}(x)$ is bounded piecewise continuous function vector, we have

$$\|\boldsymbol{v} - \boldsymbol{w}\|_{\infty,\bar{\Omega}} \le \frac{1}{\alpha}\|\boldsymbol{L}\boldsymbol{v} - \boldsymbol{L}\boldsymbol{w}\|_{\infty,\bar{\Omega}}. \tag{7}$$

Let $\bar{\Omega}^N = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ be an arbitrary nonuniform mesh, where N is a positive even discretization parameter. For $1 \le i \le N$, let $h_i = x_i - x_{i-1}$ be the local grid step and set $\hbar_i = \frac{h_i + h_{i+1}}{2}$. For a given mesh function $\{\omega_i\}_{i=0}^N$, let $D^-\omega_i = \frac{\omega_i - \omega_{i-1}}{h_i}$, $D\omega_i = \frac{\omega_{i+1} - \omega_i}{\hbar_i}$. Then, the numerical solution $\mathbf{U} = (\mathbf{U}_0, \mathbf{U}_1, \cdots, \mathbf{U}_N)^T \in \left(\mathbb{R}^{N+1}\right)^m$ is required to satisfy the following finite difference discretization of (1)-(2):

$$\mathbf{L}^N\mathbf{U}_i := -\varepsilon DD^-\mathbf{U}_i + \mathbf{a}(x_i, \mathbf{U}_i) = \mathbf{f}_i, \quad 1 \le i \le N-1, \tag{8}$$

$$\mathbf{U}_0 = \mathbf{p}, \quad \mathbf{U}_N = \mathbf{q}, \tag{9}$$

where $\mathbf{f}_i = \mathbf{f}(x_i)$, $\mathbf{U}_i = (U_{1,i}, \cdots, U_{m,i})^T$. It follows from Corollary 5 of[2] that the difference operator $\mathbf{L}^N$ satisfies the discrete maximum principle, which implies the following stability result

$$\|\mathbf{U}\|_\infty \le \max\{\|\mathbf{U}_0\|_\infty, \|\mathbf{U}_N\|_\infty\} + \max_{1\le i\le N-1} \frac{1}{\alpha}\left(\|\mathbf{a}(x_i, 0)\|_\infty + \|\mathbf{f}_i\|_\infty\right). \tag{10}$$

3. A posteriori error analysis

In this section, we will derive an a posteriori error estimation for the discrete scheme (8)-(9). Let $\widetilde{\mathbf{U}}^N(x) = \left(\widetilde{U}_1^N(x), \cdots, \widetilde{U}_m^N(x)\right)^T$, and $\widetilde{U}_j^N(x)$ be a piecewise quadratic function through points $(x_i, U_{j,i})$, $i = 0, 1, \cdots, N$, $j = 1, \cdots, m$. Then for $x \in J_i = [x_{i-1}, x_i]$, $\widetilde{\mathbf{U}}^N(x)$ can be defined as follows:

$$\widetilde{\mathbf{U}}^N(x) = \frac{1}{2}DD^-\mathbf{U}_i(x - x_{i-1})(x - x_i) + \frac{1}{h_i}\left[\mathbf{U}_i(x - x_{i-1}) + \mathbf{U}_{i-1}(x_i - x)\right], \ 1 \le i \le N-1. \tag{11}$$

For $x \in J_N$, we set $\widetilde{\mathbf{U}}^N(1) = \mathbf{q}$.

Theorem 3.1. *Let $\boldsymbol{u}(x)$ be the solution of the problem (1)-(2), $\{\boldsymbol{U}_i\}_{i=0}^N$ be the solution of the discrete scheme (8)-(9) and $\widetilde{\boldsymbol{U}}^N(x)$ be its piecewise quadratic function vector defined in (11). Then we have*

$$\max_{0\le i\le N}\|\boldsymbol{u}(x_i)-\boldsymbol{U}_i\|_\infty \le \max_{1\le i\le N-1} CQ_i, \tag{12}$$

where

$$Q_i=\max_{1\le j\le m}\left\{\left\|a_j(\cdot,\widetilde{\boldsymbol{U}}^N(\cdot))-a_j(x_i,\boldsymbol{U}_i)\right\|_{\infty,J_i}+\|f_j(\cdot)-f_j(x_i)\|_{\infty,J_i}\right\},\ i=1,\cdots,N-1. \tag{13}$$

Proof. For any $x\in J_i(1\le i\le N-1)$, we have

$$\begin{aligned}\mathbf{L}^N\widetilde{\mathbf{U}}^N(x)-\mathbf{L}\mathbf{u}(x)&=-\varepsilon DD^-\mathbf{U}_i+\mathbf{a}\left(x,\widetilde{\mathbf{U}}^N(x)\right)-\mathbf{f}(x)\\&=\mathbf{f}_i-\mathbf{a}\left(x_i,\mathbf{U}_i\right)+\mathbf{a}\left(x,\widetilde{\mathbf{U}}^N(x)\right)-\mathbf{f}(x),\end{aligned} \tag{14}$$

where we have used the discrete scheme (8).

It is easy to get $\mathbf{u}(0)=\widetilde{\mathbf{U}}^N(0)=\mathbf{p}$ from (11). Combined with $\mathbf{u}(1)=\widetilde{\mathbf{U}}^N(1)=\mathbf{q}$, it follows from Corollary 2.1.1 and (14) that

$$\begin{aligned}\left\|\mathbf{u}-\widetilde{\mathbf{U}}^N\right\|_{\infty,J_i}&\le C\left\|\mathbf{L}^N\widetilde{\mathbf{U}}^N-\mathbf{L}\mathbf{u}\right\|_{\infty,J_i}\\&\le C\left\|\mathbf{f}_i-\mathbf{a}\left(x_i,\mathbf{U}_i\right)+\mathbf{a}\left(\cdot,\widetilde{\mathbf{U}}^N(\cdot)\right)-\mathbf{f}(\cdot)\right\|_{\infty,J_i}\\&\le CQ_i.\end{aligned} \tag{15}$$

Furthermore, note that $\mathbf{u}(x_j)-\mathbf{U}_j=\mathbf{0}$, $j=0,N$, we have

$$\max_{0\le i\le N}\|\mathbf{u}(x_i)-\mathbf{U}_i\|_\infty\le\max_{1\le i\le N-1}\left\|\mathbf{u}-\widetilde{\mathbf{U}}^N\right\|_{\infty,J_i}\le CQ_i, \tag{16}$$

which completes the proof. □

4. An adaptive grid algorithm

In this section, an adaptive algorithm can be devised by using the a posteriori error estimation presented in Theorem 3.1. The mainly process of this algorithm is based on the idea by de Boor.[20] In order to adaptively design a mesh, the local contributions to the a posteriori error estimator Q_i are the same on each mesh interval. In other words, this idea is equivalent to find the points $\{(x_i,U_{j,i})\}_{i=0}^N$, $j=1,\cdots,m$, such that

$$h_i\bar{M}_i=\frac{1}{N}\sum_{k=1}^N h_i\bar{M}_i, \tag{17}$$

where $\bar{M}_i$ is a monitor function. Here, similar to [21], we choose the monitor function $\bar{M}_i=\sqrt{h_i^2+Q_i}$, $i=1,\cdots,N$, where $Q_N=Q_{N-1}$. Finally, in order to obtain an adaptive grid and the corresponding numerical solution, the following iteration algorithm is devised.

Algorithm

Step 1. For a fixed N, let $\bar{\Omega}_N^{(0)}=\left\{0,\frac{1}{N},\cdots,\frac{N-1}{N},1\right\}$ be a initial uniform mesh;

Step 2. For a given mesh $\bar{\Omega}_N^{(k)}=\left\{x_i^{(k)}\right\}_{i=0}^N$ and the corresponding computed solution $\left\{\mathbf{U}_i^{N,(k)}\right\}$. Set $\widetilde{M}_i^{(k)}=\frac{\bar{M}_{i-1}^{(k)}+\bar{M}_i^{(k)}}{2}$ for $i=1,\cdots,N$, where $\bar{M}_i^{(k)}$ is the value of monitor function calculated at the current mesh $\bar{\Omega}_N^{(k)}$, and set $\bar{M}_0^{(k)}=\bar{M}_1^{(k)}$ and $\bar{M}_N^{(k)}=\bar{M}_{N-1}^{(k)}$.

Step 3. Let $\Phi_0^{(k)}=0$ and $\Phi_i^{(k)}=\sum_{j=1}^i\widetilde{M}_j$, $i=1,\cdots,N$, for a given constant $C_0>1$, if

$$\max_{1\le i\le N}\frac{\widetilde{M}_i^{(k)}}{\Phi_N^{(k)}}\le\frac{C_0}{N} \tag{18}$$

holds true, then go to Step 5. Otherwise, continue to Step 4.

Step 4. Let $Y_i^{(k)}=i\Phi_N^{(k)}/N$ for $i=1,\cdots,N-1$ and $x=\varphi(t)$ be the piecewise linear interpolation function through knots $\left(\Phi_i^{(k)},x_i^{(k)}\right)$. Generate a new mesh $\bar{\Omega}_N^{(k+1)}=\left\{x_i^{(k+1)}\right\}_{i=0}^N$ by using equation $x_i^{(k+1)}=\varphi\left(Y_i^{(k)}\right)$, $i=1,\cdots,N-1$, where $x_0^{(k)}=0$

and $x_N^{(k)} = 1$ for $k = 0, 1, \cdots,$. Return to Step 2.

Step 5. Set $x_i^* = x_i^{(k)}$ and $\mathbf{U}_i^* = \mathbf{U}_i^{N,(k)}$ for $i = 0, 1, \cdots, N$. Stop.

5. Numerical experiments

In this section, we present some numerical results for the following system of singularly perturbed semilinear reaction-diffusion equations

$$\begin{aligned}
&-\varepsilon u_1'' + u_1 + (u_1 - 0.5u_2 - 0.5u_3)^3 = 0.6,\\
&-\varepsilon u_2'' + u_2 + (2u_2 - u_1 - u_3)^5 = 0.9,\\
&-\varepsilon u_3'' + u_3 + (u_3 - u_2)^3 = 1,\\
&u_j(0) = 0,\ u_j(1) = 0,\ j = 1, 2, 3.
\end{aligned}$$

Since the exact solution of this example is not available, the maximum point-wise errors can be evaluated as follows:

$$E_\varepsilon^N := \|\mathbf{U}_i^N - \mathbf{U}_i^{2N}\|_\infty, \tag{19}$$

where $\mathbf{U}_i^{kN} (k = 1, 2)$ is the solution of the discrete scheme (8)-(9) calculated on the final mesh $\{x_i\}_{i=0}^{kN} (k = 1, 2)$ obtained by the above grid generation algorithm, and $\mathbf{U}^{2N}(x)$ is the piece-wise linear interpolation function vector through knots $\left(x_i, \mathbf{U}_i^{2N}\right)$. For each N, we shall compute the parameter-uniform error $E^N := \max_\varepsilon E_\varepsilon^N$ over a wide range of values of ε. The rates of convergence are given by $r_\varepsilon^N = \log_2\left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}\right)$.

Note that the discrete scheme (8)-(9) is a system of nonlinear equations, we use the Newton's iteration formula to solve it with $\mathbf{0}$ as an initial vector. Here the stopping criterion is $\|\mathbf{U}^{(k)} - \mathbf{U}^{(k-1)}\|_\infty \leq 10^{-6}$, where $\mathbf{U}^{(k)}$ is the successive approximates to $\mathbf{U}$ computed iteratively, $k = 1, \cdots,$.

For $N = 64, 128, \cdots, 2048$ and $\varepsilon = 10^{-j} (j = 0, 1, \cdots, 6)$, the maximum point-wise errors and the rates of convergence obtained by using the presented adaptive grid method are listed in Table 1 as well as the average rates of convergence $\bar{r}_\varepsilon^N$. Here, we choose $C_0 = 2$. It can be seen from Table 1 that with the gradual decrease of ε, the average rates of convergence have almost reached more than 2. Meanwhile, for the purpose of comparison, the finite difference scheme (8)-(9) on a Shishkin mesh[2] is also used to solve this test problem, and the numerical results are displayed in Table 2. It is shown from these results that the accuracy of the discrete scheme (8)-(9) on a priori Shishkin mesh is almost second order, and the convergence order of the presented adaptive grid method is second order.

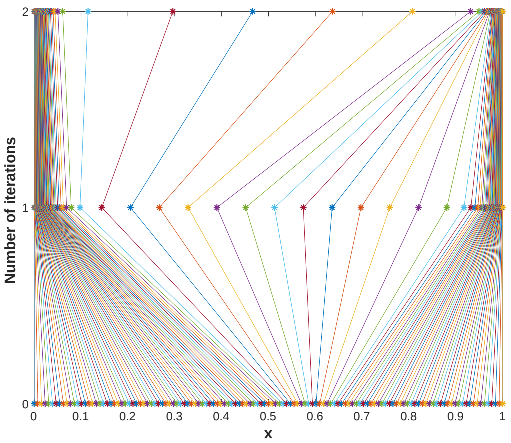
In addition, when $\varepsilon = 10^{-4}$ and $N = 128$, Figure 1(a), which should be read from bottom to top, represents the iteration process of the above adaptive grid algorithm. Meanwhile, the graph of numerical solution is also plotted in Figure 1(b). Obviously, one can see from Figure 1 that the solution of this test problem boundary layers at both ends of the interval $[0, 1]$.

Table 1. Numerical results of our adaptive grid method

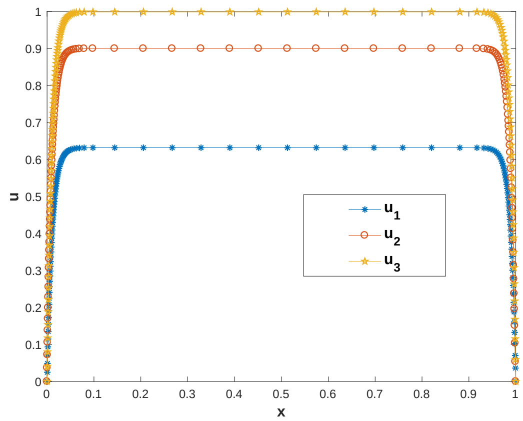
ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$	$\bar{r}_\varepsilon^N$
10^0	1.56e-06	3.91e-07	9.77e-08	2.44e-08	6.11e-09	1.53e-09	
	2.00	2.00	2.00	2.00	2.00	-	2.00
10^{-1}	4.37e-05	1.09e-05	2.73e-06	6.83e-07	1.71e-07	4.27e-08	
	2.00	2.00	2.00	2.00	2.00	-	2.00
10^{-2}	2.02e-04	5.60e-05	1.85e-05	5.75e-06	1.75e-06	5.22e-07	
	1.75	1.70	1.69	1.71	1.75	-	1.72
10^{-3}	2.69e-04	6.42e-05	1.53e-05	3.70e-06	9.10e-07	2.25e-07	
	2.07	2.07	2.05	2.03	2.01	-	2.05
10^{-4}	1.62e-03	5.75e-04	1.90e-05	4.27e-06	1.00e-06	2.40e-07	
	1.50	4.92	2.15	2.09	2.06	-	2.54
10^{-5}	1.78e-03	5.43e-04	1.76e-04	8.50e-05	1.19e-06	2.68e-07	
	1.71	1.63	1.04	6.15	2.15	-	2.54
10^{-6}	1.08e-02	7.59e-04	1.24e-04	7.41e-05	3.80e-05	3.35e-07	
	3.82	2.62	0.74	0.96	6.82	-	2.99

Table 2. Numerical results of the Shishkin grid method

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$	$N=2048$
10^{0}	1.56e-06	3.91e-07	9.77e-08	2.44e-08	6.11e-09	1.53e-09
	2.00	2.00	2.00	2.00	2.00	-
10^{-1}	4.37e-05	1.09e-05	2.73e-06	6.83e-07	1.71e-07	4.27e-08
	2.00	2.00	2.00	2.00	2.00	-
10^{-2}	2.80e-04	7.02e-05	1.75e-05	4.39e-06	1.10e-06	2.75e-07
	1.99	2.00	2.00	2.00	2.00	-
10^{-3}	2.73e-03	6.96e-04	1.75e-04	4.37e-05	1.10e-05	2.74e-06
	1.97	1.99	2.00	2.00	2.00	-
10^{-4}	6.33e-03	2.31e-03	7.69e-04	2.45e-04	7.57e-05	2.29e-05
	1.45	1.59	1.65	1.69	1.72	-
10^{-5}	6.33e-03	2.31e-03	7.69e-04	2.45e-04	7.57e-05	2.29e-05
	1.45	1.59	1.65	1.69	1.72	-
10^{-6}	6.33e-03	2.31e-03	7.69e-04	2.45e-04	7.57e-05	2.29e-05
	1.45	1.59	1.65	1.69	1.72	-



(a) Evolution of the mesh



(b) numerical solution

Fig. 1. Evolution of mesh and comparison between numerical and exact solution for Example 2.

6. Conclusions

This paper mainly develop an adaptive grid method for a system of singularly perturbed reaction-diffusion equations. Based on the quadratic polynomial interpolation, we derive an a posteriori error estimation, which is used to construct an adaptive grid algorithm. It should be pointed out that our presented adaptive grid method can be extended the other singularly perturbed reaction-diffusion problems.

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Conflicts of Interest

The authors declare no conflict of interest.

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