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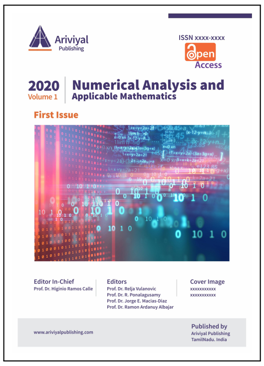

Difference Scheme on a Non-Uniform Mesh for Singularly Perturbed Reaction Diffusion Equations with Integral Boundary Condition

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Abstract: In this paper we consider singularly perturbed reaction diffusion equations with integral boundary condition. A numerical method based on finite difference scheme on Shishkin mesh is presented. This method is proved to be almost second order convergent. An error estimate is derived in the discrete norm. Numerical results are presented, which validate the theoretical results.

Keywords: Singular perturbation problems; Finite difference scheme; Shishkin mesh; Integral boundary condition; Error estimate

1. Introduction

Differential equations with a small parameter ε $(0 < \varepsilon \le 1)$ multiplying the highest order derivatives are called singularly perturbed differential equations. Classical numerical methods are inappropriate for singularly perturbed problems because the solutions of such equations have steep gradients in narrow layer regions of the domain. Therefore, it is important to develop suitable methods to these problems, whose accuracy does not depend on the parameter ε, that is, methods that are convergent ε-uniformly.[21, 8] One of the simplest ways to derive such methods consists of using a class of piecewise uniform meshes (Shishkin mesh, Bakhvalov mesh), which are constructed a priori in function of sizes of parameter ε, the problem data, and the number of corresponding mesh points.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. A class of boundary value problems with integral boundary conditions have plenty of applications such as in electro chemistry [9] thermo elasticity,[10] heat conduction[5] etc. The authors of[19, 12, 1] have proved the existence, uniqueness and stability of differential equations with integral boundary conditions. Boundary value problems involving integral boundary conditions have received considerable attention in recent years.[14, 13, 6, 3]

So far ordinary differential equations with integral boundary conditions are considered. In the following singularly perturbed ordinary differential equations with integral boundary conditions will be discussed.

In[11, 4] the authors have considered the following singularly perturbed boundary value problem of the form

$$\varepsilon^2 u''(x) + \varepsilon a(x)u'(x) - b(x)u(x) = f(x),\ 0 < x < l, \tag{1}$$

$$u(0) = \mu_0, \tag{2}$$

$$u(l) - \int_{l_0}^{l_1} g(x)u(x)dx = \mu_l,\ 0 \le l_0 < l_1 \le l. \tag{3}$$

Here μ_0 and μ_l are given constants, $a(x),\ b(x),\ f(x)$ and $g(x)$ are smooth functions in $[0, l]$. They suggested the method of integral identities using exponential basis functions and interpolating quadrature rules with the weight and remainder term in integral form an exponentially fitted difference scheme on an uniform mesh is developed which is shown to be ε- uniformly first order accurate in the discrete maximum norm.

In[2, 20, 7] the authors have considered the following singular perturbation problem with integral boundary condition

$$\varepsilon u'(x) + f(t, u) = 0,\ t \in I = (0, T],\ T > 0, \tag{4}$$

$$u(0) = \mu u(T) + \int_0^T b(s)u(s)ds + d, \tag{5}$$

where μ and d are given constants. In fact, the authors of[2] suggested a uniform finite difference scheme on piecewise uniform

Shishkin meshes for the above problem and also proved that, this method has almost first order convergence. In[20] the method of boundary layer function and the Banach fixedpoint theorem, the uniformly valid asymptotic solution of the above problem is obtained. Zhongdi Cen and Xin Cai[7] proposed a hybrid finite difference scheme on Shishkin mesh.

Mustafa Kudu and Gabil M. Amiraliyev[18] have presented a fitted finite difference method for solving the following singularly perturbed problem with integral boundary condition

$$\varepsilon u''(x) + a(x)u'(x) = f(x),\ 0 < x < l, \tag{6}$$

$$u'(0) = \frac{\mu_0}{\varepsilon}, \tag{7}$$

$$\int_0^l d(x)u(x)dx = \mu_1, \tag{8}$$

where $\mu_i (i = 0, 1)$ are given constants.

In[15, 16, 17] the authors have considered the following parameterized singular perturbation problem with integral boundary condition:

$$\varepsilon u'(t) + f(t, u, \lambda) = 0,\ t \in (0, T),\ T > 0 \tag{9}$$

$$u(0) + \int_0^T c(s)u(s)ds = A \tag{10}$$

$$u(T) = B \tag{11}$$

λ is known as the control parameter, A and B are given constants. In fact, in[15] asymptotic estimates for the solution and its first derivative have been established. In[17] Mustfa kudu et al. a numerical algorithm based on upwind finite difference operator and an appropriate piecewise uniform mesh is constructed for the above problem. In[16] a uniform finite difference method is suggested on a Bakhvalov mesh to solve the above problem.

V. Raja and A. Tamilselvan[23, 22] considered singularly perturbed convection diffusion equations with integral boundary conditions and they suggested finite difference scheme on equidistant mesh.

This paper is organized as follows. In section 2, the statement of the problem is given. In section 3, some preliminary results like maximum principle, stability and bounds on the derivatives of the solution for the continuous problem are discussed. The numerical method is described in section 4. In section 5 the error analysis for approximate solution is presented. In section 6 numerical results are given. The conclusion is presented in section 7.

2. Statement of the problem

Motivated by the works of, [11, 4, 2, 18, 15, 23, 22, 17] we consider the following singularly perturbed problem with integral boundary condition:

$$Lu = -\varepsilon u''(x) + b(x)u(x) = f(x), \qquad 0 < x < 1, \tag{12}$$

$$u(0) = A \tag{13}$$

$$L_1 u(1) = u(1) - \varepsilon \int_0^1 g(x)u(x)dx = B \tag{14}$$

where $0 < \varepsilon << 1$ is a small positive parameter, A and B are given constants, $b(x) \geq \beta > 0$, $x \in \bar{\Omega} = [0, 1]$, $g(x)$ is nonnegative with $\int_0^1 g(x)dx < 1$ and $f(x), b(x)$ are sufficiently smooth functions on $\bar{\Omega}$. The solution $u(x)$ of problem $(12) - (14)$ has in general boundary layers at $x = 0$ and $x = 1$.

Through out the paper, we assume that $\sqrt{\varepsilon} \leq CN^{-1}$, C denotes a generic positive constant. The supremum norm is used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem: $\|u\|_D = \sup_{x \in D} |u(x)|$.

3. The continuous problem

We first establish priori bounds for the solution and its derivatives. The differential operator L satisfies the following maximum principle.

Theorem 3.1. ***(Maximum Principle)*** *Let $\psi(x) \in C^2(\bar{\Omega})$ be any function satisfying $\psi(0) \geq 0$, $L_1\psi(1) \geq 0$ and $L\psi(x) \geq 0$, $\forall x \in \Omega$. Then $\psi(x) \geq 0$, $\forall\ x \in \bar{\Omega}$.*

Proof. Define

$$s(x) = x + 1. \tag{15}$$

Note that $s(x) > 0, \forall x \in \bar{\Omega}$, $Ls(x) > 0, \forall x \in \Omega$, $s(0) > 0$ and $L_1 s(1) > 0$.

Let $\mu = \max\{\frac{-\psi(x)}{s(x)} : x \in \bar{\Omega}\}$. Then there exists $x_0 \in \bar{\Omega}$ such that $\psi(x_0) + \mu s(x_0) = 0$ and $\psi(x) + \mu s(x) \geq 0, \forall x \in \bar{\Omega}$. Therefore, the function $(\psi + \mu s)$ attains its minimum at $x = x_0$. Suppose the theorem does not hold true, then $\mu > 0$.

Case (i): $x_0 = 0$

$$0 < (\psi + \mu s)(0) = \psi(0) + \mu s(0) = 0.$$

It is a contradiction.

Case (ii): $x_0 \in \Omega$

$$0 < L(\psi + \mu s)(x_0) = -\varepsilon(\psi + \mu s)''(x_0) + a(x_0)(\psi + \mu s)(x_0) \leq 0.$$

It is a contradiction.

Case (iii): $x_0 = 1$

$$0 < L_1(\psi + \mu s)(1) = (\psi + \mu s)(1) - \varepsilon \int_0^1 g(x)(\psi + \mu s)(x)dx \leq 0.$$

It is a contradiction. Hence the proof of the theorem. □

Corollary 3.1.1. *(Stability Result) The solution $u(x)$ of problem $(12) - (14)$ satisfies the bound*

$$|u(x)| \leq C \max\{|u(0)|, |L_1 u(1)|, ||f||\}, \forall x \in \bar{\Omega}. \tag{16}$$

Proof. Let $C > 0$ be a constant. Define $\psi^{\pm}(x) = CMs(x) \pm u(x), x \in \bar{\Omega}$, where $M = \max\{|u(0)|, |L_1 u(1)|, ||f||\}$ and s is a test function defined by (15).

Further

$$\begin{aligned} \psi^{\pm}(0) &= CMs(0) \pm u(0) > 0, \\ L_1 \psi^{\pm}(1) &= CMs(1) \pm u(1) - CM\varepsilon \int_0^1 g(x)s(x)dx \pm \varepsilon \int_0^1 g(x)u(x)dx \geq 0 \end{aligned}$$

and

$$\begin{aligned} L\psi^{\pm}(x) &= MCb(x)s(x) \pm f(x) \\ &\geq M\beta C \pm f(x) \geq 0 \end{aligned}$$

by a proper choice of C. Then by maximum principle, we have $\psi^{\pm}(x) \geq 0, x \in \bar{\Omega}$. Therefore, $|u(x)| \leq C \max\{|u(0)|, |L_1 u(1)|, ||f||\}$, $\forall x \in \bar{\Omega}$. □

Bounds for the derivatives of $u(x)$ are given in the following lemma.

Lemma 3.2. *Let $u(x)$ be the solution of $(12) - (14)$. Then, for $1 \leq k \leq 4$,*

$$\|u^{(k)}\|_{\bar{\Omega}} \leq C(1 + \varepsilon^{-k/2}).$$

Proof. Using corollary 3.1.1 and applying arguments as given in [21] this lemma gets proved. □

To derive error estimates, we decompose the solution $u(x)$ into smooth and singular components as

$$u(x) = v(x) + w(x).$$

Here $v = v_0 + \varepsilon v_1$, where v_0 is the solution of the reduced problem, w is the solution of the homogeneous problem

$$Lw = 0, \qquad w(0) = A - v_0(0), \qquad L_1 w(1) = B - L_1 v_0(1) \tag{17}$$

and, consequently v_1 satisfies

$$Lv_1 = v_0'', \qquad v_1(0) = 0, \qquad L_1 v_1(1) = 0.$$

Because of the bound on v_0'', it is clear that v_1 is the solution of a problem similar to $(12) - (14)$. This implies that, for $0 \leq k \leq 4$,

$$|v_1^k(x)| \leq C(1 + \varepsilon^{-k/2}).$$

Decompose the singular component as

$$w = w_L + w_R,$$

where the boundary layer functions w_L and w_R are defined to be solutions of the problems

$$\begin{aligned} Lw_L &= 0, \quad w_L(0) = w(0), \quad L_1 w_L(1) = 0, \\ Lw_R &= 0, \quad w_R(0) = 0, \quad L_1 w_R(1) = L_1 w(1). \end{aligned}$$

In the following lemma, we derive bounds for the components of the solution and their respective derivatives.

Lemma 3.3. *The regular component v and the singular components w_L and w_R of the solution $u(x)$ of the problem $(12)-(14)$ satisfy the following bounds,*

$$\begin{aligned} |v^{(k)}(x)| &\leq C(1+\varepsilon^{-(k-2)/2}) \\ |w_L^{(k)}(x)| &\leq C\varepsilon^{-k/2}(\varepsilon + e^{-x\sqrt{\frac{\beta}{\varepsilon}}}) \\ |w_R^{(k)}(x)| &\leq C\varepsilon^{-k/2}e^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}}, \quad \textit{for} \quad 0 \leq k \leq 4, \quad \forall x \in \bar{\Omega}. \end{aligned}$$

Proof. Note that $v = v_0 + \varepsilon v_1$, the proof for the bounds on v is an immediate consequence of the above estimates $v_0^{(k)}, v_1^{(k)}$.

Introduce the functions $\psi^{\pm}(x) = C\varepsilon(1 + e^{-\sqrt{\frac{\beta}{\varepsilon}}} - e^{-x\sqrt{\frac{\beta}{\varepsilon}}}) \pm w_L(x)$ where the constant C is chosen sufficiently large. Then $\psi^{\pm}(0) \geq 0$,

$$\begin{aligned} L_1\psi^{\pm}(1) &= \psi(1) - \varepsilon \int_0^1 g(x)\psi(x)dx \\ &\geq C(1 - \varepsilon \int_0^1 g(x)dx) \pm L_1 w_L(1) \geq 0 \end{aligned}$$

and

$$L\psi^{\pm}(1) \geq Cb(x)\varepsilon(1 - e^{-x\sqrt{\frac{\beta}{\varepsilon}}}) \geq 0.$$

Hence the maximum principle gives $\psi^{\pm}(x) \geq 0$ and so

$$|w_L(x)| \leq C(\varepsilon + e^{-x\sqrt{\frac{\beta}{\varepsilon}}}), \quad \text{for all } \ x \in \bar{\Omega}.$$

Bounds on the derivatives of w_L are established in [21]. By an appropriate choice of barrier function, the bound on the component of w_R is given by

$$|w_R(x)| \leq Ce^{-(1-x)\sqrt{\frac{\beta}{\varepsilon}}} \quad \text{for all } \ x \in \bar{\Omega}.$$

Analogous arguments are used to establish the bounds on the derivatives of w_R. □

4. The discrete problem

On $\bar{\Omega}$ a piecewise uniform Shiskin mesh of N (≥ 4) mesh intervals is constructed. The domain $\bar{\Omega}$ is partitioned into three subintervals $[0,\sigma]$, $[\sigma, 1-\sigma]$ and $[1-\sigma, 1]$ where σ is the transition parameter defined by $\sigma = \min\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}}\ln N\}$. On $[0,\sigma]$ and $[1-\sigma,1]$ a uniform mesh with $\frac{N}{4}$ mesh intervals are placed, while $[\sigma, 1-\sigma]$ has a uniform mesh with $\frac{N}{2}$ mesh intervals. The interior mesh points are denoted by Ω^N. Let $h_i = x_i - x_{i-1}$ be the mesh step and $\bar{h_i} = \frac{h_{i+1}+h_i}{2}$.

The discrete problem corresponding to $(12)-(14)$ is:
Find U such that

$$L^N U = -\varepsilon\delta^2 U(x_i) + b(x_i)U(x_i) = f(x_i), \forall x_i \in \Omega^N. \tag{18}$$

$$U(x_0) = A, \tag{19}$$

$$L_1^N U(x_N) = U(x_N) - \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}U_{i-1} + g_i U_i}{2} h_i = B, \forall x_i \in \Omega^N. \tag{20}$$

where

$$\begin{aligned}\delta^2 Z_i &= \frac{1}{\bar{h}_i}\left(\frac{Z_{i+1}-Z_i}{h_{i+1}} - \frac{Z_i - Z_{i-1}}{h_i}\right),\\ g_i &= g(x_i),\ U_i = U(x_i).\end{aligned}$$

Theorem 4.1. ***(Discrete maximum principle)*** *Assume that the mesh function Ψ_i satisfies $\Psi_0 \geq 0$, $L_1^N\Psi(1) \geq 0$ and $L^N\Psi_i \geq 0$, for all i, $1 \leq i \leq N-1$. Then $\Psi_i \geq 0$, for all i, $0 \leq i \leq N$.*

Proof. Define

$$S(x_i) = x_i + 1. \tag{21}$$

Note that $S(x_i) > 0, \forall x_i \in \bar{\Omega}^N$, $L^N S(x_i) > 0, \forall x \in \Omega^N$, $S(0) > 0$ and $L_1^N S(1) > 0$.

Let $\mu = \max\{\frac{-\Psi(x_i)}{S(x_i)} : x_i \in \bar{\Omega}^N\}$. Then there exists $x_0 \in \bar{\Omega}_\tau^N$ such that $\Psi(x_0) + \mu S(x_0) = 0$ and $\Psi(x_i) + \mu S(x_i) \geq 0, \forall x_i \in \bar{\Omega}^N$. Therefore, the function $(\Psi + \mu S)$ attains its minimum at $x_i = x_0$.

Suppose the theorem does not hold true,then $\mu > 0$.

Case (i): $x_0 = 0$

$$0 < (\Psi + \mu S)(0) = \Psi(0) + \mu S(0) = 0$$

It is a contradiction.

Case (ii): $x_0 \in \Omega^N$

$$\begin{aligned}0 < L^N(\Psi + \mu S)(x_0) &= -\varepsilon\delta^2(\Psi+\mu S)(x_0) + b_i(\Psi+\mu S)(x_0)\\ &= \frac{-\varepsilon}{\bar{h}_i}\Big(\frac{(\Psi+\mu S)(x_{0+1}) - (\Psi+\mu S)(x_0)}{h_{i+1}} -\\ &\quad \frac{(\Psi+\mu S)(x_0) - (\Psi+\mu S)(x_{0-1})}{h_i}\Big) \leq 0\end{aligned}$$

It is a contradiction because

$(\Psi + \mu S)(x_{0+1}) - (\Psi + \mu S)(x_0) \geq 0$ and $(\Psi + \mu S)(x_0) - (\Psi + \mu S)(x_{0-1)} \leq 0$

Case (iii): $x_0 = 1$

$$\begin{aligned}0 &< L_1^N(\Psi+\mu S)(1) = (\Psi+\mu S)(1) -\\ &\quad \varepsilon\sum_{i=1}^{N}\frac{(\Psi+\mu S)(x_{i-1})g(x_{i-1}) + (\Psi+\mu S)(x_i)g(x_i)}{2}h_i \leq 0.\end{aligned}$$

It is a contraction. Hence the proof of the theorem. □

Lemma 4.2. ***(Stability Result)*** *If Φ_i is any function then*

$$|\Phi_i| \leq C\max\left(\sup_{1\leq j\leq N-1}|L^N\Phi_j|, |\Phi(0)|, |L_1^N\Phi(1)|\right), \qquad for \quad 0 \leq i \leq N. \tag{22}$$

Proof. Let $C > 0$ be a constant. Define $\Psi_i^\pm = CMS_i \pm \Phi_i$, where

$M = \max(\sup_{1\leq j\leq N-1}|L^N\Phi_j|, |\Phi(0)|, |L_1^N\Phi(1)|)$ and S is a test function defined by (21). Then

$$\begin{aligned}\Psi^\pm(0) &= CMS(0) \pm \Phi(0) > 0.\\ L_1^N\Psi^\pm(1) &= \Psi^\pm(1) - \varepsilon\sum_{i=1}^{N}\frac{\Psi_{i-1}^\pm g_{i-1} + \Psi_i^\pm g_i}{2}h_i\\ &= CMS(1) \pm \Phi(1) - \varepsilon\sum_{i=1}^{N}\frac{(CMS_{i-1}\pm\Phi_{i-1})g_{i-1} + (CMS_i \pm \Phi_i)g_i}{2}h_i\\ &= CML_1^N S(1) \pm L_1^N\Phi(1) \geq 0,\end{aligned}$$

and

$$\begin{aligned}L^N\Psi_i^\pm &= -\varepsilon\delta^2\Psi_i^\pm + b_i\Psi_i^\pm\\ &= -\varepsilon\delta^2(CMS_i \pm \Phi_i) + b_i(CMS_i \pm \Phi_i)\\ &= CML^N S_i \pm L^N\Phi_i \geq 0.\end{aligned}$$

by a proper choice of C. Then by discrete maximum principle, we have $\Psi_i^\pm \geq 0$, for $0 \leq i \leq N$, as required. □

Analogous to the continuous case, the discrete solution U can be decomposed as

$$U = V + W,$$

where V and W are respectively, the solutions of the problems

$$L^N V = f(x_i), \quad x_i \in \Omega^N, \quad V(0) = v(0), \quad L_1^N V(1) = L_1 v(1) \tag{23}$$

$$L^N W = 0, \quad x_i \in \Omega^N, \quad W(0) = w(0), \quad L_1^N W(1) = L_1 w_(1). \tag{24}$$

The singular component W is decomposed as

$$W = W_L + W_R,$$

where W_L and W_R are defined by

$$L^N W_L = 0, \quad W_L(0) = w_L(0), \quad L_1^N W_L(1) = 0, \tag{25}$$

$$L^N W_R = 0, \quad W_R(0) = 0, \quad L_1^N W_R(1) = L_1 W_R(1). \tag{26}$$

5. Error estimates for the solution

We obtain separate error estimates for each component of the numerical solution.

Lemma 5.1. *The error in the smooth component of the numerical solution is bounded as*

$$|(V - v)(x_i)| \le CN^{-2}, \textit{ for all} \quad x_i \in \bar{\Omega}^N,$$

where v is the solution of regular component of the original problem and V is the solution of (23).

Proof. Consider the local truncation error

$$\begin{aligned} L^N(V - v) &= L^N V - L^N v = f - L^N v \\ &= Lv - L^N v \end{aligned}$$

By[21] we have

$$|L^N(V - v)(x_i)| \le \begin{cases} C\sqrt{\varepsilon}N^{-1}, & x_i \in \{\sigma, 1-\sigma\}, \\ CN^{-2}, & otherwise. \end{cases}$$

Therefore

$$|L^N(V - v)(x_i)| \le CN^{-2}.$$

Further

$$\begin{aligned} L_1^N(V - v)(x_N) &= L_1^N V(x_N) - L_1^N v(x_N) \\ &= B - L_1^N v(x_N) \\ &= L_1 v(x_N) - L_1^N v(x_N) \\ |L_1^N(V - v)(x_N)| &\le C\varepsilon(h_1^3 v''(\chi_1) + \cdots + h_N^3 v''(\chi_N)) \\ &\le CN^{-2} \end{aligned}$$

where $x_{i-1} \le \chi_i \le x_i$, $1 \le i \le N$.

Applying lemma 4.2 we have $|(V - v)(x_i)| \le CN^{-2}$, for all $\quad x_i \in \bar{\Omega}^N$. □

Lemma 5.2. *The error in the singular component of the numerical solution is bounded as*

$$|(W - w)(x_i)| \le CN^{-2}\ln^2 N, \textit{ for all } x_i \in \bar{\Omega}^N$$

where w is the solution of (17) *and W is the solution of* (24).

Proof. The error can be written in the form

$$W - w = (W_L - w_L) + (W_R - w_R),$$

and the errors $W_L - w_L$ and $W_R - w_R$ are estimated separately.

First the estimate for $W_L - w_L$ is given. The argument depends on whether $\sigma = \frac{1}{4}$ or $\sigma = 2\sqrt{\frac{\varepsilon}{\beta}} \ln N < \frac{1}{4}$

Case(i): $\sigma = \frac{1}{4}$

In this case the mesh is uniform and $2\sqrt{\frac{\varepsilon}{\beta}} \ln N \geq \frac{1}{4}$, it is clear that $x_i - x_{i-1} = N^{-1}$ and $\varepsilon^{-\frac{1}{2}} \leq C \ln N$. By [21],

$$|L^N(W_L - w_L)(x_i)| \quad \leq \quad C(N^{-1} \ln N)^2$$

and

$$\begin{aligned}
L_1^N(W_L - w_L)(x_N) &= L_1^N W_L(x_N) - L_1^N w_L(x_N) \\
&= B - L_1^N w_L(x_N) \\
&= L_1 w_L(x_N) - L_1^N w_L(x_N) \\
|L_1^N(W_L - w_L)(x_N)| &\leq \varepsilon|C(h_1^3 w''(\chi_1) + \cdots + h_N^3 w''(\chi_N))| \\
&\leq CN^{-2}
\end{aligned}$$

where $x_{i-1} \leq \chi_i \leq x_i$. Applying lemma 4.2 to the function $(W_L - w_L)(x_i)$ gives

$$|(W_L - w_L)(x_i)| \leq C(N^{-1} \ln N)^2. \tag{27}$$

Case(ii): $\sigma < \frac{1}{4}$

The mesh is piecewise uniform, with the mesh spacing $2(1-2\sigma)/N$ in the subinterval $[\sigma, 1-\sigma]$ and $4\sigma/N$ in each of the subintervals $[0, \sigma]$ and $[1 - \sigma, 1]$. The argument depends on the mesh spacing. By [21],

$$|L^N(W_L - w_L)(x_i)| \quad \leq \quad C(N^{-1} \ln N)^2 \quad \text{for all} \quad x_i \in (0, 1)$$

and

$$\begin{aligned}
|L_1^N(W_L - w_L)(x_N)| &\leq \varepsilon|C(h_1^3 w''(\chi_1) + \cdots + h_N^3 w''(\chi_N))| \\
&\leq C(h_1^3 + \cdots + h_N^3) \\
&\leq CN^{-2}
\end{aligned}$$

where $x_{i-1} \leq \chi_i \leq x_i$, an application of lemma 4.2 to the function $(W_L - w_L)(x_i)$ gives

$$|(W_L - w_L)(x_i)| \leq CN^{-2} \ln^2 N.$$

Analogous arguments are used to establish the error estimate for W_R. This completes the proof. □

The above error estimates for the individual components of the numerical solution now lead to the following theorem on the error estimate for the numerical solution U, which is obtained by combining them using the triangle inequality.

Theorem 5.3. *If u is the solution of $(12) - (14)$ and U is the corresponding numerical solution of $(18) - (20)$, then we have*

$$\|U - u\|_{\bar{\Omega}^N} \leq CN^{-2} \ln^2 N.$$

Proof. Combining Lemma 5.1 and Lemma 5.2, the proof gets completed. □

6. Numerical Results

Example 6.1.

$$-\varepsilon u''(x) + u(x) = 1, \qquad 0 < x < 1 \tag{28}$$

with boundary conditions

$$u(0) = 0, \tag{29}$$

$$u(1) - \varepsilon \int_0^1 \frac{x}{2} u(x)dx = 0. \tag{30}$$

Table 1. Maximum pointwise errors and order of convergence for Example 6.1

	Number of mesh points N						
ε	64	128	256	512	1024	2048	4096
2^{0}	5.439e-06	1.360e-06	3.399e-07	8.499e-08	2.125e-08	5.312e-09	1.328e-09
2^{-2}	2.153e-05	5.384e-06	1.346e-06	3.365e-07	8.412e-08	2.103e-08	5.259e-09
2^{-4}	8.352e-05	2.088e-05	5.221e-06	1.305e-06	3.263e-07	8.159e-08	2.039e-08
2^{-6}	2.425e-04	6.071e-05	1.518e-05	3.796e-06	9.490e-07	2.372e-07	5.931e-08
2^{-8}	9.525e-04	2.391e-04	5.985e-05	1.496e-05	3.742e-06	9.355e-07	2.338e-07
2^{-10}	3.747e-03	9.525e-04	2.391e-04	5.985e-05	1.496e-05	3.742e-06	9.355e-07
2^{-12}	4.041e-03	1.391e-03	4.586e-04	1.454e-04	4.492e-05	1.359e-05	3.742e-06
2^{-14}	4.041e-03	1.391e-03	4.586e-04	1.454e-04	4.492e-05	1.359e-05	4.045e-06
2^{-16}	4.041e-03	1.391e-03	4.586e-04	1.454e-04	4.492e-05	1.359e-05	4.045e-06
2^{-18}	4.041e-03	1.391e-03	4.586e-04	1.454e-04	4.492e-05	1.359e-05	4.045e-06
E^N	4.041e-03	1.391e-03	4.586e-04	1.454e-04	4.492e-05	1.359e-05	4.045e-06
R^N	1.538e+00	1.601e+00	1.656e+00	1.695e+00	1.724e+00	1.748e+00	-

Table 2. Maximum pointwise errors and order of convergence for Example 6.2

	Number of mesh points N						
ε	64	128	256	512	1024	2048	4096
2^{0}	5.177e-06	1.294e-06	3.236e-07	8.091e-08	2.022e-08	5.056e-09	1.264e-09
2^{-2}	1.435e-05	3.590e-06	8.976e-07	2.244e-07	5.610e-08	1.402e-08	3.506e-09
2^{-4}	4.568e-05	1.147e-05	2.870e-06	7.176e-07	1.794e-07	4.485e-08	1.121e-08
2^{-6}	1.785e-04	4.504e-05	1.131e-05	2.829e-06	7.074e-07	1.768e-07	4.421e-08
2^{-8}	1.822e-04	1.778e-04	4.488e-05	1.126e-05	2.818e-06	7.048e-07	1.762e-07
2^{-10}	7.204e-04	3.138e-04	1.249e-04	4.940e-05	1.682e-05	5.612e-06	1.699e-06
2^{-12}	7.192e-04	3.135e-04	1.247e-04	4.937e-05	1.681e-05	5.611e-06	1.761e-06
2^{-14}	7.185e-04	3.133e-04	1.247e-04	4.935e-05	1.681e-05	5.610e-06	1.761e-06
2^{-16}	7.182e-04	3.132e-04	1.246e-04	4.935e-05	1.681e-05	5.610e-06	1.761e-06
2^{-18}	7.181e-04	3.132e-04	1.246e-04	4.934e-05	1.681e-05	5.609e-06	1.761e-06
D^N	7.215e-04	3.141e-04	1.250e-04	4.940e-05	1.682e-05	5.612e-06	1.761e-06
P^N	1.199e+00	1.329e+00	1.339e+00	1.554e+00	1.583e+00	1.671e+00	-

Its exact solution is given by

$$u = \frac{(\varepsilon - 2\varepsilon^2 - 4 + 4e^{\frac{-1}{\sqrt{\varepsilon}}}(1 + \frac{\varepsilon^{3/2}}{2} + \frac{\varepsilon^2}{2}))e^{\frac{x}{\sqrt{\varepsilon}}}}{4e^{\frac{1}{\sqrt{\varepsilon}}}(1 - \frac{\varepsilon^{3/2}}{2} + \frac{\varepsilon^2}{2}) - 4e^{\frac{-1}{\sqrt{\varepsilon}}}(1 + \frac{\varepsilon^{3/2}}{2} + \frac{\varepsilon^2}{2})} + \frac{(2\varepsilon^2 - \varepsilon + 4 - 4e^{\frac{1}{\sqrt{\varepsilon}}}(1 - \frac{\varepsilon^{3/2}}{2} + \frac{\varepsilon^2}{2}))e^{\frac{-x}{\sqrt{\varepsilon}}}}{4e^{\frac{1}{\sqrt{\varepsilon}}}(1 - \frac{\varepsilon^{3/2}}{2} + \frac{\varepsilon^2}{2}) - 4e^{\frac{-1}{\sqrt{\varepsilon}}}(1 + \frac{\varepsilon^{3/2}}{2} + \frac{\varepsilon^2}{2})} + 1.$$

Since the problem has an analytical solution, for every ε the maximum pointwise errors are estimated by

$$E_\varepsilon^N = \max_{x_i \in \bar{\Omega}} |u(x_i) - U^N(x_i)| \quad \text{and} \quad E^N = \max_{\varepsilon} E_\varepsilon^N$$

where U^N denotes the numerical solution. The order of convergence is obtained by

$$R_\varepsilon^N = \log_2(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}) \text{ and } R^N = \log_2(\frac{E^N}{E^{2N}}).$$

Example 6.2.

$$-\varepsilon u''(x) + (5 + x)u(x) = 1, \qquad 0 < x < 1 \tag{31}$$

with boundary conditions

$$u(0) = 0, \tag{32}$$

$$u(1) - \varepsilon \int_0^1 \frac{x}{2} u(x)dx = 0. \tag{33}$$

The exact solution of the Example 6.2 is not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. Define the double mesh difference to be

$$D_\varepsilon^N = \max_{x_i \in \bar{\Omega}^N} |U^N(x_i) - \bar{U}^{2N}(x_i)| \text{ and } D^N = \max_{\varepsilon} D_\varepsilon^N$$

where $\bar{U}^{2N}(x_i)$ is the piecewise linear interpolant of the mesh function $U^{2N}(x_i)$ onto [0,1]. From these quantities the order of convergence is computed from

$$P_\varepsilon^N = \log_2(\frac{D_\varepsilon^N}{D_\varepsilon^{2N}}) \text{ and } P^N = \log_2(\frac{D^N}{D^{2N}}).$$

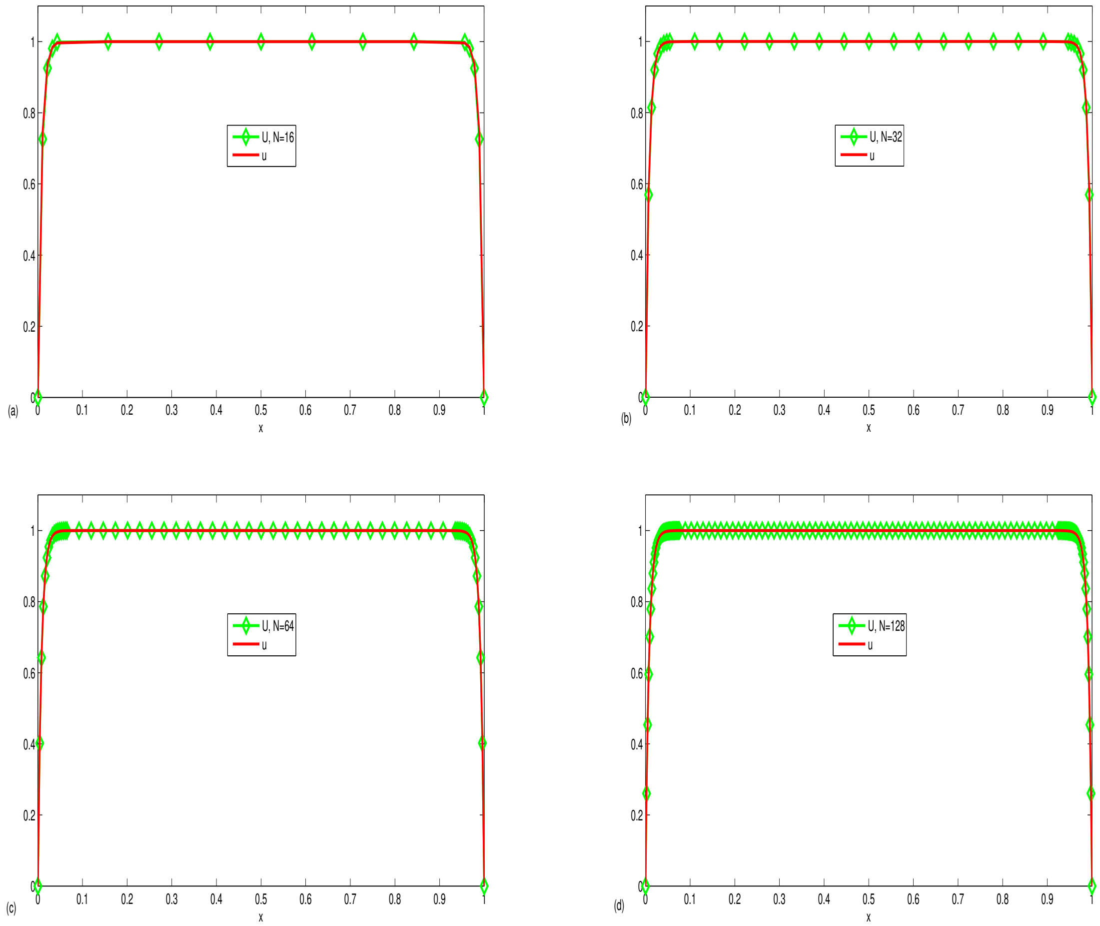

Fig. 1. Exact and approximate solutions of Example 6.1 for $\varepsilon = 2^{-14}$.

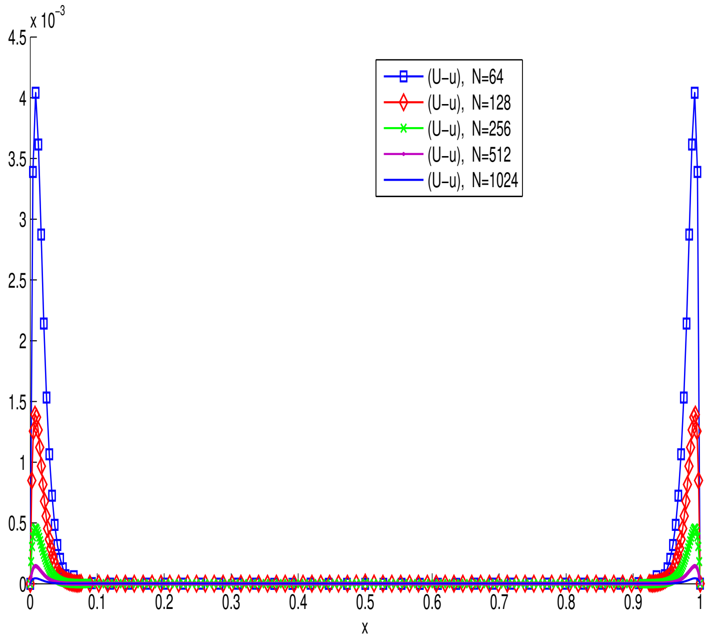

Fig. 2. The error in the numerical solution with respect to the exact solution of Example 6.1 for $\varepsilon = 2^{-14}$.

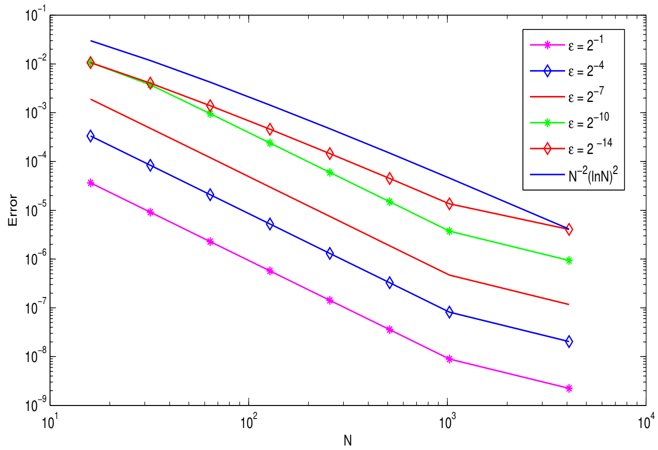

Fig. 3. Loglog plot of the maximum pointwise errors for Example 6.1

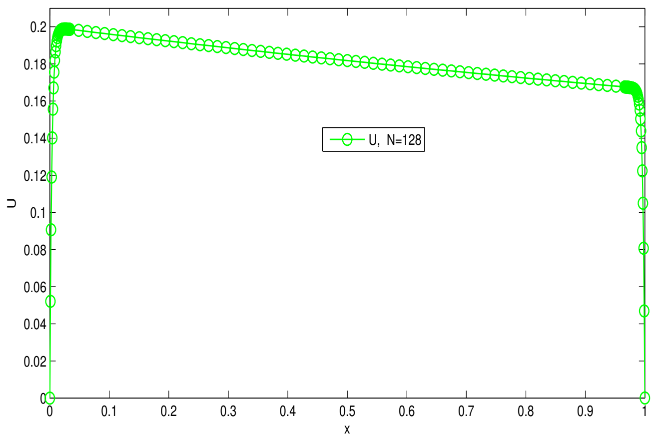

Fig. 4. Graph of the numerical solution of Example 6.2 for $\varepsilon = 2^{-14}$.

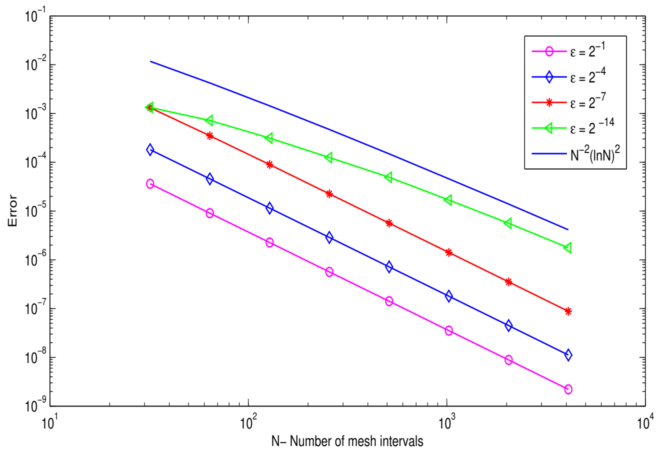

Fig. 5. Loglog plot of the maximum pointwise errors for Example 6.2

7. Conclusion

We have solved the singularly perturbed boundary value problem $(12) - (14)$ with integral boundary condition, using finite difference method on Shishkin mesh. The method is shown to be of order $O(N^{-2}\ln^2 N)$, that is, the method has almost second order convergence with respect to ε. Two examples are given to illustrate the numerical method. Our numerical results reflect the theoretical estimates. Maximum pointwise errors and order of convergence of the Examples 6.1 and 6.2 are given in Table 1 and 2 respectively. Comparison of exact and approximate solutions of Example 6.1 for $\varepsilon = 2^{-14}$ with various N values are given in Figure 1. The error between exact and numerical solutions of Example 6.1 for various values of N is given in Figure 2. Loglog plot of the maximum pointwise errors of Example 6.1 is given in Figure 3. The numerical solution of Example 6.2 is plotted for $\varepsilon = 2^{-14}$ and $N = 128$ in Figure 4. The maximum pointwise errors for Example 6.2 through loglolg plot is presented in Figure 5.

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Conflicts of Interest

The authors declare no conflict of interest.

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