

Fitted Numerical Method for Singularly Perturbed Delay Differential Equations

Gemechis File Duressa*^a and Habtamu Garoma Debela^a

^aDepartment of Mathematics, College of Natural Sciences, Jimma University, Jimma, Ethiopia.

*Corresponding author E-mail address: gammeef@gmail.com (Gemechis File Duressa)

ISSN: 2582-8274



Publication details

Received: 17th May 2020
Revised: 14th June 2020
Accepted: 15th June 2020
Published: 26th June 2020

Abstract: This paper presents a numerical method to solve singularly perturbed delay differential equations. The solution of this problem exhibits layer or oscillatory behaviour depending on the sign of the sum of the coefficients in reaction terms. A fourth order fitted numerical scheme on uniform mesh is developed. The stability and convergence of the proposed method have been established. The effect of delay parameter (small shift) on the boundary layer(s) has also been analyzed and depicted in graphs. The applicability of the proposed scheme is validated by implementing it on four model examples. Maximum absolute errors in comparison with the other numerical experiments are tabulated to illustrate the proposed method.

Keywords: Singular perturbation; Delay differential equation; Fitted operator; Twin layers; Oscillatory layers

1. Introduction

Differential equations with a small parameter $0 < \varepsilon \ll 1$ multiplying the highest order derivative are called singularly perturbed differential equations. Mathematically, any differential equation in which the highest derivative is multiplied by a small positive parameter ε and containing at least one delay/advance parameter, is known as a singularly perturbed differential-difference equation.^[1] Such type of equations arises frequently from the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex,^[2] the study of bi stable devices,^[3] and vibrational problems in control theory,^[4] etc. When perturbation parameter ε is very small, most numerical methods for solving such problems may be unstable and give inaccurate results. So, it is important to develop suitable numerical methods to solve singularly perturbed delay differential equations.

Hence, in the recent times, many researchers have been trying to develop numerical methods for solving singularly perturbed delay differential equations. For example, the authors in^[5] proposed computational method of first order for singularly perturbed delay reaction-diffusion equations with layer or oscillatory behaviour. Authors in^[6] presented fourth order finite difference scheme for second order singularly perturbed differential-difference equation with negative shift. Authors of^[7] presented exponentially fitted second order finite difference scheme for a class of singularly perturbed delay differential equations with large delay. In^[8] the numerical solution of singularly perturbed differential-difference equations with dual layer was presented. Recently, the authors in^[9] presented computational method for solving singularly perturbed delay differential equation with twin layers or oscillatory behaviour. But, still there is a lack of accuracy because of the treatment of singularly perturbed problems is not trivial and the solution depends on perturbation parameter ε and mesh size h .^[10-12] Due to this, numerical treatment of singularly perturbed delay differential equations needs improvement. Therefore, it is important to develop more accurate and convergent numerical method for solving singularly perturbed delay differential equations.

Thus, the purpose of this study is to develop stable, convergent and more accurate numerical method for solving singularly perturbed delay differential equations.

2. Description of the method

Consider the following singularly perturbed delay differential equation of the form:

$$\varepsilon^2 y'' + a(x)y(x-\delta) + b(x)y(x) = f(x), \quad 0 < x < 1 \quad (1)$$

subject to the interval and boundary conditions,

$$y(x) = \phi(x), \quad \text{for } -\delta \leq x \leq 0 \text{ and } y(1) = \varphi \quad (2)$$

where ε is perturbation parameter, $0 < \varepsilon \ll 1$ and δ is a small delay parameter of $O(\varepsilon)$, $0 < \delta \ll 1$, $a(x), b(x), f(x)$ and $\phi(x)$ are bounded smooth functions and φ is a given constant. The layer or oscillatory behaviour of the problem under consideration is maintained for $\delta \neq 0$ but sufficiently small, depending on the sign of $a(x) + b(x)$, for all $x \in (0, 1)$. If $a(x) + b(x) < 0$, the solution of the problem in equations (1) and (2) exhibits layer behaviour and if $a(x) + b(x) > 0$, it exhibits oscillatory behaviour. Therefore, if the solution exhibits layer behaviour, there will be two boundary layers which will occur at both end points $x = 0$ and $x = 1$.^[12]

By using Taylor series expansion in the neighborhood of x , we have:

$$y(x-\delta) \approx y(x) - \delta y'(x) + O(\delta^2) \quad (3)$$

Substituting equation (3) into equation (1), we obtain an asymptotically equivalent singularly perturbed two point boundary value problem:

$$\varepsilon^2 y''(x) + p(x)y'(x) + q(x)y(x) = f(x) \quad (4)$$

Where, $p(x) = -\delta a(x)$, $q(x) = a(x) + b(x)$

under boundary conditions,

$$y(0) = \phi_0, \quad y(1) = \varphi \quad (5)$$

Discretizing the given interval $[0, 1]$ in to N equal parts with constant mesh size h , we have $x_i = x_0 + ih$, $i = 0, 1, 2, \dots, N$.

Using the Taylor's expansions of y_{i+1} and y_{i-1} up to $O(h^5)$, we get the finite difference approximations for y'_i and y''_i as

$$y'_i = \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y'''_i + \tau_1 \quad (6)$$

where $\tau_1 = -\frac{h^4}{20} y^{(5)}(\xi_1)$, for $\xi_1 \in [x_{i-1}, x_i]$, and

$$y''_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}_i + \tau_2 \quad (7)$$

where $\tau_2 = -\frac{h^4}{360} y^{(6)}(\xi_2)$, for $\xi_2 \in [x_{i-1}, x_i]$.

Substituting equations (6) and (7) in to equation (4) and simplifying, we obtain:

$$\frac{\varepsilon^2}{h^2} (y_{i+1} - 2y_i + y_{i-1}) + \frac{p_i}{2h} (y_{i+1} - y_{i-1}) - \frac{h^2}{6} p_i y'_i - \frac{\varepsilon^2 h^2}{12} y^{(4)}_i + q_i y_i = f_i + \tau_i \quad (8)$$

where $\tau_i = h^4 \left(\frac{p_i}{120} y^{(5)}(\xi_1) + \frac{\varepsilon^2}{360} y^{(6)}(\xi_2) \right) + O(h^5)$ is the local truncation error and $p(x_i) = p_i$, $q(x_i) = q_i$, $f(x_i) = f_i$ and $y_i \approx y(x_i)$.

By successively differentiating both sides of equation (4) and evaluating at x_i , and using into equation (8), we obtain:

$$\frac{\varepsilon^2}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{p_i}{2h}(y_{i+1} - y_{i-1}) + \varepsilon^2 \left\{ \frac{h^2 p_i^2}{6\varepsilon^2} + \frac{h^2}{6} \left(2p'_i + q_i - \frac{p_i^2}{\varepsilon^2} \right) \right\} y_i'' \tag{9}$$

$$+ A_i y'_i + (B_i + b_i) y_i = f_i + C_i,$$

where,

$$\begin{cases} A_i = \frac{h^2}{6\varepsilon^2} p_i(p'_i + q_i) - \frac{h^2}{12} \left(\frac{p_i}{\varepsilon^2} (p'_i + q_i) - p_i'' - 2q'_i \right) \\ B_i = \frac{h^2}{6\varepsilon^2} p_i q'_i - \frac{h^2}{12} \left(\frac{p_i q'_i}{\varepsilon^2} - q_i'' \right) \\ C_i = f_i + \frac{a_i h^2}{12\varepsilon^2} f'_i + \frac{h^2}{12} f_i'' + \tau_i \end{cases}$$

Now, introducing a fitting parameter σ and using central difference approximation for y_i'' and y'_i in equation (9), we have

$$\frac{\sigma \varepsilon^2}{h^2} \left\{ 1 + \frac{h^2 p_i^2}{12\varepsilon^2} + \frac{h^2}{12} (2p'_i + q_i) \right\} (y_{i+1} - 2y_i + y_{i-1}) + \frac{1}{2h} (p_i + A_i) (y_{i+1} - y_{i-1}) \tag{10}$$

$$+ (B_i + q_i) y_i = f_i + C_i$$

Multiplying both sides of equation (10) by h and taking the limit as $h \rightarrow 0$, we obtain:

$$\frac{\sigma}{12\rho} (12 + \rho^2 p_i^2) \lim_{h \rightarrow 0} \{ y_{i+1} - 2y_i + y_{i-1} \} + \frac{p_i}{2} \lim_{h \rightarrow 0} \{ y_{i+1} - y_{i-1} \} = 0 \tag{11}$$

where $\rho = \frac{h}{\varepsilon^2}$.

From the theory of singular perturbations and ^[13], we have two cases for $p(x) > 0$ and $p(x) < 0$.

Case 1: For $p(x) < 0$ (right-end boundary layer), we have:

$$\begin{cases} \lim_{h \rightarrow 0} (y_{i+1} - 2y_i + y_{i-1}) = (\phi_0 - y_0(0)) e^{-p(0) \left(\frac{-1+i\rho}{\varepsilon^2} \right)} \left(e^{-p(0)\rho} + e^{p(0)\rho} - 2 \right) \\ \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) = (\phi_0 - y_0(0)) e^{-p(0) \left(\frac{-1+i\rho}{\varepsilon^2} \right)} \left(e^{-p(0)\rho} - e^{p(0)\rho} \right) \end{cases}$$

Thus, from equation (11), we get:

$$\sigma(0) = \frac{6\rho p(0)}{(12 + \rho^2 p^2(0))} \coth \left(\frac{p(0)\rho}{2} \right)$$

Case 2: For $p(x) > 0$ (left-end boundary layer), we have:

$$\begin{cases} \lim_{h \rightarrow 0} (y_{i+1} - 2y_i + y_{i-1}) = (\phi_0 - y_0(1)) e^{-p(1) \left(\frac{-1+i\rho}{\varepsilon^2} \right)} \left(e^{-p(1)\rho} + e^{p(1)\rho} - 2 \right) \\ \lim_{h \rightarrow 0} (y_{i+1} - y_{i-1}) = (\phi_0 - y_0(1)) e^{-p(1) \left(\frac{-1+i\rho}{\varepsilon^2} \right)} \left(e^{-p(1)\rho} - e^{p(1)\rho} \right) \end{cases}$$

Thus, from equation (11), we get:

$$\sigma(1) = \frac{6\rho p(1)}{(12 + \rho^2 p^2(1))} \coth \left(\frac{p(1)\rho}{2} \right)$$

In general, for discretization, we take a variable fitting parameter as:

$$\sigma_i = \frac{6\rho_i p_i}{(12 + (\rho_i p_i)^2)} \coth \left(\frac{p_i \rho_i}{2} \right) \tag{12}$$

where $\rho_i = \frac{h}{\varepsilon^2}$.

Simplifying equation (10), we get the tri-diagonal system of the equation of the form

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \text{ for } i = 1, 2, \dots, N-1. \tag{13}$$

where

$$\begin{cases} E_i = \frac{\sigma_i \varepsilon^2}{h^2} + \frac{\sigma_i p_i^2}{12} + \frac{\sigma_i \varepsilon^2}{12} (2p'_i + q_i) - \frac{1}{2h} (p_i + A_i) \\ F_i = \frac{2\sigma_i \varepsilon^2}{h^2} + \frac{\sigma_i p_i^2}{6} + \frac{\sigma_i \varepsilon^2}{6} (2p'_i + q_i) - p_i - B_i \\ G_i = \frac{\sigma_i \varepsilon^2}{h^2} + \frac{\sigma_i p_i^2}{12} + \frac{\sigma_i \varepsilon^2}{12} (2p'_i + q_i) + \frac{1}{2h} (p_i + A_i) \\ H_i = f_i + C_i \end{cases}$$

The tri-diagonal system in equation (13) can be easily solved by Thomas algorithm with help of Matlab 2013.

3. Stability and convergence analysis

Case 1: Layer Behavior (i.e. $a(x) + b(x) < 0$, for $x \in (0,1)$). Thus $q(x) < 0$, since $\varepsilon^2 > 0$)

Lemma 1: (Continuous minimum principle) If $y(0) \geq 0$ and $Ly(x) \leq 0$, for all $x \in (0,1)$, then the solution $y(x) \geq 0$ for all $x \in [0,1]$ for equations (4) and (5).

Proof: Suppose $t \in (0,1)$, such that $y(t) = \min_{x \in (0,1)} y(x)$ and $y(t) < 0$. since $t \notin \{0,1\}$ and is a point of minima, then $y'(t) = 0$ and $y''(t) \geq 0$.

Therefore, we have

$Ly(t) = \varepsilon^2 y''(t) + p(t)y'(t) + q(t)y(t) > 0$. Since $y(t) < 0$ (by assumption) and $q(t) < 0$. But this is a contradiction. Then, it follows that $y(x) \geq 0$ for $x \in [0,1]$.

Theorem 1: If the solution of the problem in equations (4) and (5) satisfies

$$|y(x)| \leq K \max \left\{ |y(0)|, \max_{x \in (0,1)} |Ly(x)| \right\}$$

for some constant $K \geq 1$, then the solution is stable.

Proof: We define two functions, $\psi^\pm(x) = K \max \left\{ |y(0)|, \max_{x \in (0,1)} |Ly(x)| \right\} \pm y(x)$. Then we get $\psi^\pm(0) \geq 0$ and

$$L\psi^\pm(x) = Kq(x) \max \left\{ |y(0)|, \max_{x \in (0,1)} |Ly(x)| \right\} \pm Ly(x) \leq 0,$$

since $q(x) < 0$ and for suitable choice of K . Therefore, by lemma 1, we get, $\psi^\pm(x) \geq 0$, for all $x \in (0,1)$.

$$\text{So, } |y(x)| \leq K \max \left\{ |y(0)|, \max_{x \in (0,1)} |Ly(x)| \right\}.$$

Hence, the stability of the solutions of the problem in equations (4) and (5) is proved for the case of the layer behavior.

Lemma 2: The finite difference operator L^N in equation (13) satisfies the discrete minimum principle. i.e. if w_i is any mesh function such that $w_0 \geq 0$ and $L^N w_i \leq 0$, for all $x_i \in (0,1)$, then $w_i \geq 0$ for all $x \in [0,1]$.

Proof: Suppose there exists a positive integer K such that $w_k < 0$ and $w_i = \min_{0 \leq i \leq N} w_i$. Then, from equation (13) we have

$$\begin{aligned} L^N w_k &= E_k w_{k-1} - F_k w_k + G_k w_{k+1} \\ &= \left\{ \frac{\sigma_k \varepsilon^2}{h^2} + \frac{\sigma_k p_k^2}{12} + \frac{\sigma_k \varepsilon^2}{12} (2p'_k + q_k) \right\} (w_{k+1} - 2w_k + w_{k-1}) + \frac{1}{2h} (p_k + A_k) (w_{k+1} - w_{k-1}) + (p_k + B_k) w_k \\ &= \left\{ \frac{\sigma_k \varepsilon^2}{h^2} + \frac{\sigma_k p_k^2}{12} + \frac{\sigma_k \varepsilon^2}{12} (2p'_k + q_k) \right\} (w_{i+1} - w_k) + \left\{ \frac{\sigma_k \varepsilon^2}{h^2} + \frac{\sigma_k p_k^2}{12} + \frac{\sigma_k \varepsilon^2}{12} (2p'_k + q_k) \right\} (w_{k-1} - w_k) \\ &\quad + \frac{1}{2h} \left(p_k + \left(\frac{h^2}{6\varepsilon^2} p_k (p'_k + q_k) - \frac{h^2}{12} \left(\frac{p_k}{\varepsilon^2} (p'_k + q_k) - p''_k - 2q'_k \right) \right) \right) (w_{i+1} - w_{i-1}) \\ &\quad + \left(p_k + \left(\frac{h^2}{6\varepsilon^2} p_i q'_i - \frac{h^2}{12} \left(\frac{p_i q'_i}{\varepsilon^2} - q''_i \right) \right) \right) w_k \end{aligned}$$

For sufficiently small h and for suitable value of p_k , we obtain $L^N w_k > 0$. Since, $w_k < 0$ (by the assumption) and $B_k \rightarrow p_k < 0$. But, this is a contradiction. Hence $w_i \geq 0$ for all $x_i \in [0, 1]$.

Theorem 2: The finite difference operator L^N in equation (13) is stable for $a(x) + b(x) < 0$, if w_i is any mesh function, then

$$|w_i| \leq K \max \left\{ |w_0|, \max_{x_i \in (0,1)} |Lw_i| \right\}, \text{ for some constant } K \geq 1.$$

Proof: We define two functions, $\psi^\pm = K \max \left\{ |w(0)|, \max_{x \in (0,1)} |Lw_i| \right\} \pm w_i$. Then, similar to Theorem 1, we get $\psi^\pm(0) \geq 0$ and

$$L\psi^\pm = Kq_i \max \left\{ |w_0|, \max_{x \in (0,1)} |Lw_i| \right\} \pm Lw_i \leq 0,$$

since $a_i + b_i < 0 \Rightarrow q(x) < 0$ and $K \geq 1$. Therefore, by Lemma 2 we get, $\psi_i^\pm \geq 0$, for all $x_i \in (0,1)$. $\Rightarrow \psi_i^\pm \equiv C \max \left\{ |w_0|, \max_{x_i \in (0,1)} |Lw_i| \right\} \pm w_i \geq 0$.

$$\text{Thus, } |w_i| \leq C \max \left\{ |w_0|, \max_{x_i \in (0,1)} |Lw_i| \right\}$$

This proves the stability of the scheme for the case of layer behavior.

Case 2: Oscillatory behavior (i.e. $a(x) + b(x) > 0$, for $x \in (0,1)$. Thus $q(x) > 0$, since $\varepsilon^2 > 0$)

The continuous maximum principle and stability of the solution of equation (4) and (5) are presented as follows for the case of oscillatory behavior.

Lemma3: If $y(0) \geq 0$ and $Ly(x) \leq 0$, for all $x \in (0,1)$, then the solution $y(x) \geq 0$ for all $x \in [0,1]$ for equations (4) and (5).

Proof: Suppose $t \in (0,1)$, such that $y(t) = \max_{x \in (0,1)} y(x)$ and $y(t) < 0$. Since $t \notin \{0,1\}$ and is a point of maxima, therefore $y'(t) = 0$ and $y''(t) \leq 0$.

Therefore, we have

$Ly(t) = \varepsilon^2 y''(t) + p(t)y'(t) + q(t)y(t) < 0$, since $y(t) < 0$ (by assumption) and $q(t) > 0$. But this is a contradiction. Hence, $y(x) \geq 0$ for $x \in [0,1]$.

Theorem 3: If the solution of the problem in equations (4) and (5) satisfies

$$|y(x)| \leq K \max \left\{ |y(0)|, \max_{x \in (0,1)} |Ly(x)| \right\}$$

for some constant $k \geq 1$.

Proof: The proof is analogous to Theorem 1. Hence, the stability of the solutions of the problem in equations (4) and (5) is proved for the case of oscillatory behavior. Now, we present the stability of the discrete problem in equation (13) for the case of oscillatory behavior.

Lemma 4: The finite difference operator L^N in equation (13) satisfies the discrete maximum principle, if w_i is any mesh function such that $w_0 \geq 0$ and $L^N w_i \geq 0$, for all $x_i \in (0,1)$, then $w_i \geq 0$ for all $x \in [0,1]$.

Proof: Suppose there exists a positive integer k such that $w_k < 0$ and $w_i = \max_{0 \leq i \leq N} w_i$. Then, from equation (13) we have

$$\begin{aligned} L^N w_k &= E_k w_{k-1} - F_k w_k + G_k w_{k+1} \\ &= \left\{ \frac{\sigma_k \varepsilon^2}{h^2} + \frac{\sigma_k p_k^2}{12} + \frac{\sigma_k \varepsilon^2}{12} (2p'_k + q_k) \right\} (w_{k+1} - 2w_k + w_{k-1}) + \frac{1}{2h} (p_k + A_k) (w_{k+1} - w_{k-1}) + (p_k + B_k) w_k \\ &= \left\{ \frac{\sigma_k \varepsilon^2}{h^2} + \frac{\sigma_k p_k^2}{12} + \frac{\sigma_k \varepsilon^2}{12} (2p'_k + q_k) \right\} (w_{i+1} - w_k) + \left\{ \frac{\sigma_k \varepsilon^2}{h^2} + \frac{\sigma_k p_k^2}{12} + \frac{\sigma_k \varepsilon^2}{12} (2p'_k + q_k) \right\} (w_{k-1} - w_k) \\ &\quad + \frac{1}{2h} \left(p_k + \left(\frac{h^2}{6\varepsilon^2} p_k (p'_k + q_k) - \frac{h^2}{12} \left(\frac{p_k}{\varepsilon^2} (p'_k + q_k) - p''_k - 2q'_k \right) \right) \right) (w_{i+1} - w_{i-1}) \\ &\quad + \left(p_k + \left(\frac{h^2}{6\varepsilon^2} p_i q'_i - \frac{h^2}{12} \left(\frac{p_i q'_i}{\varepsilon^2} - q''_i \right) \right) \right) w_k \end{aligned}$$

For sufficiently small h and for suitable value of p_k , we obtain $L^N w_k < 0$. Since, $w_k < 0$ (by the assumption) and $B_k \rightarrow p_k > 0$. But, this is a contradiction. Hence $w_i \geq 0$ for all $x_i \in (0,1)$.

Theorem 4: The finite difference operator L^N in equation (13) is stable for $a(x) + b(x) > 0$, (i.e. $q(x) > 0$), if W_i is any mesh function, then

$$|w_i| \leq K \max \left\{ |w_0|, \max_{x_i \in (0,1)} |Lw_i| \right\}, \text{ for some constant } K \geq 1.$$

Proof: The proof is similar to Theorem 2. This proves the scheme for the case of oscillatory behavior.

Definition (Uniform Convergence): Let y be a solution of equations (1) and (2). Consider a difference scheme for solving equations (1) and (2). If the scheme has a numerical solution y^N that satisfies $\|y - y^N\| \leq Ch^p$, where $C > 0$ and $p > 0$ are independent of ε and of mesh size h , then we say the scheme uniformly convergent to y with respect to the norm $\|\cdot\|$.^[14]

Lemma 5: The bound for derivative of the solution $y(x)$ of the problem (1)-(3) when $x \in \Omega_1 = (0,1)$ is given by

$$\begin{cases} |y^{(k)}(x)| \leq C \left(1 + \varepsilon^{-2k} \exp\left(\frac{-\alpha x_j}{\varepsilon}\right) \right), & 0 \leq k \leq 4, \quad j = 1, 2, \dots, N-1. \\ |y^{(k)}(x)| \leq C \left(1 + \varepsilon^{-2k} \exp\left(\frac{-\alpha(1-x_j)}{\varepsilon}\right) \right), & 0 \leq k \leq 4, \quad j = 1, 2, \dots, N-1. \end{cases}$$

Proof: For the proof refer^[15]

For any mesh function z_i , define the following difference operators

$$D^+ z_i = \frac{z_{i+1} - z_i}{h}, \quad D^- z_i = \frac{z_i - z_{i-1}}{h}, \quad D^0 z_i = \frac{z_{i+1} - z_{i-1}}{2h}, \quad D^+ D^- z_i = \frac{z_{i+1} - 2z_i + z_{i-1}}{h^2}, \tag{14}$$

Theorem 5: Let $y(x_i)$ and y_i be respectively the analytical solution of equations (4) and (5) and numerical solutions of equation (13). Then for sufficiently large N , the following parameter uniform error estimate holds:

$$\sup_{0 < \varepsilon \leq 1} \|y(x_i) - y_i\| \leq C^* N^{-4} \tag{15}$$

Proof: Let us consider the local truncation error defined as

$$L^N(y(x_i) - y_i) = \varepsilon^2 \sigma(\rho) \left(\frac{d^2}{dx^2} - D^+ D^- + \frac{h^2}{12} \frac{d^4}{dx^4} \right) y(x_i) + p(x_i) \left(\frac{d}{dx} - D^0 + \frac{h^2}{6} \frac{d^2}{dx^2} \right) y(x_i) \tag{16}$$

where $\varepsilon^2 \sigma(\rho) = \varepsilon^2 \left(\frac{6\rho p(0)}{(12 + \rho^2 p^2(0))} \coth\left(\frac{p(0)\rho}{2}\right) \right) \approx p(0) \frac{N^{-1}}{2} \coth\left(p(0) \frac{N^{-1}}{2\varepsilon^2}\right)$ since $\rho = \frac{N^{-1}}{\varepsilon^2}$. In our assumption $\varepsilon^2 \leq h = N^{-1}$.

By considering N is fixed and taking the limit for $\varepsilon^2 \rightarrow 0$, we obtain the following

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \sigma(\rho) = \lim_{\varepsilon \rightarrow 0} p(0) \frac{N^{-1}}{2} \coth\left(p(0) \frac{N^{-1}}{2\varepsilon^2}\right) = CN^{-1}$$

From Taylor series expansion, the bound for the difference becomes

$$\begin{cases} \left\| \left(\frac{d^2}{dx^2} - D^+ D^- \right) y(x_i) \right\| \leq CN^{-3} \left\| \frac{d^4(y(x_i))}{dx^4} \right\| \\ \left\| \left(\frac{d}{dx} - D^0 \right) y(x_i) \right\| \leq CN^{-2} \left\| \frac{d^3(y(x_i))}{dx^3} \right\| \end{cases}$$

where $\left\| \frac{d^k(y(x_i))}{dx^k} \right\| = \sup_{x_i \in (x_0, x_N)} \left(\frac{d^k y(x_i)}{dx^k} \right)$, $k = 3, 4$.

Now using the bounds and the assumption $\varepsilon^2 \leq N^{-1}$, equation (16) reduces to

$$\begin{aligned} \|L^N(y(x_i) - y_i)\| &= \left\| \varepsilon^2 \sigma(\rho) \left(\frac{d^2}{dx^2} - D^+ D^- \right) y(x_i) + \varepsilon^2 \sigma(\rho) \frac{h^2}{12} y_i^{(4)} + p(x_i) \left(\frac{d}{dx} - D^0 \right) y(x_i) + \frac{h^2}{6} y_i^{(2)} \right\| \\ &\leq \left\| \varepsilon^2 \sigma(\rho) \left(\frac{d^2}{dx^2} - D^+ D^- \right) y(x_i) \right\| + \left\| \varepsilon^2 \sigma(\rho) \frac{h^2}{12} y_i^{(4)} \right\| + \left\| p(x_i) \left(\frac{d}{dx} - D^0 \right) y(x_i) \right\| + \left\| \frac{h^2}{6} y_i^{(2)} \right\| \\ &\leq CN^{-3} \left\| \frac{d^4(y(x_i))}{dx^4} \right\| + CN^{-2} \left\| \frac{d^3(y(x_i))}{dx^3} \right\| + CN^{-2} \left\| \frac{d^2(y(x_i))}{dx^2} \right\| \end{aligned} \tag{17}$$

Here, the target is to show the scheme convergence independent on the number of mesh points.

By using the bounds for the derivatives of the right layer solution in Lemma 5, we obtain

$$\begin{aligned} \|L^N(y(x_i) - y_i)\| &\leq CN^{-3} \left\| \frac{d^4(y(x_i))}{dx^4} \right\| + CN^{-2} \left\| \frac{d^3(y(x_i))}{dx^3} \right\| + CN^{-2} \left\| \frac{d^2(y(x_i))}{dx^2} \right\| \\ &\leq CN^{-3} \left(1 + \varepsilon^{-8} \exp\left(\frac{-\alpha(1-x_j)}{\varepsilon^2}\right) \right) + CN^{-2} \left(1 + \varepsilon^{-6} \exp\left(\frac{-\alpha(1-x_j)}{\varepsilon^2}\right) \right) \\ &\quad + CN^{-2} \left(1 + \varepsilon^{-4} \exp\left(\frac{-\alpha(1-x_j)}{\varepsilon^2}\right) \right) \\ &\leq CN^{-4} \left(1 + \varepsilon^{-8} \exp\left(\frac{-\alpha(1-x_j)}{\varepsilon^2}\right) \right), \text{ since } \varepsilon^{-8} \geq \varepsilon^{-6} \geq \varepsilon^{-4} \end{aligned} \tag{18}$$

Lemma 6: For a fixed mesh and for $\varepsilon^2 \rightarrow 0$, it holds

$$\lim_{\varepsilon^2 \rightarrow 0} \max_{1 \leq j \leq N-1} \frac{\exp\left(\frac{-\alpha(1-x_j)}{\varepsilon^2}\right)}{\varepsilon^{2m}} = 0, \quad m = 1, 2, 3, \dots \tag{19}$$

Proof: Refer from [16]

By using Lemma 6 into equation (18), results to

$$\|L^h(y(x_i) - y_i)\| \leq CN^{-4} \tag{20}$$

Hence, by discrete maximum principle, we obtain

$$\|y(x_i) - y_i\| \leq CN^{-4} \tag{21}$$

Thus, result of equation (21) shows equation (15). Hence the proof.

Remark: A similar analysis for convergence may be carried out for the left layer case.

4. Numerical Examples and Results

To demonstrate the applicability of the method, we implemented the method on four numerical examples, two with twin boundary layers and two with oscillatory behavior. Since those examples have no exact solution, the numerical solutions are computed using double mesh principle. The maximum absolute errors are computed using double mesh principle given by

$$Z_h = \max_i |y_i^h - y_i^{h/2}|, \quad i = 1, 2, \dots, N-1 \tag{22}$$

where y_i^h is the numerical solution on the mesh $\{x_i\}_1^N$ at the nodal point x_i and $x_i = x_0 + ih, i = 1, 2, \dots, N-1$ and $y_i^{h/2}$ is the numerical solution on a mesh, obtained by bisecting the original mesh with N number of intervals.^[9]

Example 1: Consider the singularly perturbed delay reaction-diffusion equation with layer behavior,

$$\varepsilon^2 y''(x) - 2y(x - \delta) - y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \quad -\delta \leq x \leq 0, \quad y(1) = 0$$

The maximum absolute errors are presented in Tables 1 and 2 for different values of ε and δ . The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in the Fig. 1.

Table 1. The maximum absolute errors of Example 1, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	N = 100	N = 200	N = 300	N = 400	N = 500
Present Method					
0.03	5.4036e-07	3.3830e-08	6.6845e-09	2.1156e-09	8.6674e-10
0.05	2.1884e-07	1.3703e-08	2.7080e-09	8.5688e-10	3.5100e-10
0.09	1.6321e-06	1.0240e-07	2.0186e-08	6.3913e-09	2.6166e-09
Method in [12]					
0.03	3.3211e-06	2.0765e-07	4.1021e-08	1.2982e-08	5.3183e-09
0.05	3.3265e-06	2.0860e-07	4.1236e-08	1.3045e-08	5.3441e-09
0.09	3.9208e-06	2.4744e-07	4.8834e-08	1.5467e-08	6.3334e-09
Method in [5]					
0.03	3.2676e-03	1.6475e-03	1.1015e-03	8.2735e-04	6.6245e-04
0.05	3.2657e-03	1.6526e-03	1.1062e-03	8.3136e-04	6.6593e-04
0.09	3.5460e-03	1.7987e-03	1.2051e-03	9.0609e-04	7.2594e-04

Table 2. The maximum absolute errors of Example 1, for different values of ε with $\delta = 0.03$.

$\varepsilon \downarrow$	N = 100	N = 200	N = 300	N = 400	N = 500
Present Method					
2^{-1}	1.4376e-09	8.9855e-11	1.7756e-11	5.6259e-12	2.2786e-12
2^{-2}	2.0175e-08	1.2614e-09	2.4919e-10	7.8851e-11	3.2296e-11
2^{-3}	2.6054e-07	1.6306e-08	3.2231e-09	1.0199e-09	4.1772e-10
2^{-4}	1.4351e-06	8.9975e-08	1.7805e-08	5.6356e-09	2.3084e-09
2^{-5}	2.2448e-04	1.3296e-05	2.5999e-06	8.1969e-07	3.3519e-09
Method in [12]					
2^{-1}	6.2873e-09	3.9296e-10	7.7611e-11	2.4565e-11	1.0067e-11
2^{-2}	9.1057e-08	5.6925e-09	1.1245e-09	3.5581e-10	1.4575e-10
2^{-3}	1.3770e-06	8.6125e-08	1.7022e-08	5.3861e-09	2.2060e-09
2^{-4}	2.1676e-05	1.3569e-06	2.6910e-07	8.5189e-08	3.4888e-08
2^{-5}	4.2477e-04	2.6670e-05	5.2734e-06	1.6692e-06	6.8381e-07
Method in [5]					
2^{-1}	9.2363e-04	4.6407e-04	3.0991e-04	2.3263e-04	1.8619e-04
2^{-2}	1.6390e-03	8.2516e-04	5.5141e-04	4.1404e-04	3.3146e-04
2^{-3}	2.7044e-03	1.3653e-03	9.1315e-04	6.8602e-04	5.4937e-04
2^{-4}	4.1751e-03	2.1168e-03	1.4178e-03	1.0658e-03	8.5380e-04
2^{-5}	6.2518e-03	3.1866e-03	2.1382e-03	1.6088e-03	1.2895e-03

Example 2: Consider the singularly perturbed delay reaction-diffusion equation with layer behavior,

$$\varepsilon^2 y''(x) + 0.25y(x - \delta) - y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \quad -\delta \leq x \leq 0, \quad y(1) = 0$$

The maximum absolute errors are presented in Tables 3 and 4 for different values of ε and δ . The graph of the computed solution for $\varepsilon = 0.01$ and different values of δ is also given in the Fig. 2.

Table 3. The maximum absolute errors of Example 2, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	N = 100	N = 200	N = 300	N = 400	N = 500
Present Method					
0.03	9.7795e-08	6.1136e-09	1.2077e-09	3.8212e-10	1.5652e-10
0.05	9.8920e-08	6.1882e-09	1.2224e-09	3.8679e-10	1.5844e-10
0.09	9.9722e-08	6.2342e-09	1.2315e-09	3.8969e-10	1.5962e-10
Method in [12]					
0.03	4.2909e-07	2.6820e-08	5.2979e-09	1.6763e-09	6.8659e-10
0.05	4.4265e-07	2.7686e-08	5.4687e-09	1.7304e-09	7.0877e-10
0.09	4.7354e-07	2.9598e-08	5.8468e-09	1.8501e-09	7.5778e-10
Method in [5]					
0.03	2.1226e-03	1.0639e-03	7.0985e-04	5.3259e-04	4.2617e-04
0.05	2.1099e-03	1.0574e-03	7.0543e-04	5.2928e-04	4.2351e-04
0.09	2.0816e-03	1.0426e-03	6.9547e-04	5.2178e-04	4.1750e-04

Table 4. The maximum absolute errors of Example 2, for different values of ε with $\delta = 0.03$.

$\varepsilon \downarrow$	N = 100	N = 200	N = 300	N = 400	N = 500
Present Method					
2^{-1}	1.4036e-10	8.7691e-12	1.7249e-12	5.4062e-13	1.9877e-13
2^{-2}	3.0052e-09	1.8783e-10	3.7101e-11	1.1698e-11	4.8391e-12
2^{-3}	4.0374e-08	2.5238e-09	4.9861e-10	1.5775e-10	6.4608e-11
2^{-4}	6.4568e-07	4.0384e-08	7.9815e-09	2.5256e-09	1.0345e-09
2^{-5}	1.0399e-05	6.5163e-07	1.2878e-07	4.0766e-08	1.6703e-08
Method in [12]					
2^{-1}	6.8226e-10	3.8033e-11	7.5153e-12	2.3742e-12	9.5853e-13
2^{-2}	1.3051e-08	8.1568e-10	1.6112e-10	5.0953e-11	2.0934e-11
2^{-3}	1.7639e-07	1.1025e-08	2.1780e-09	6.8914e-10	2.8226e-10
2^{-4}	2.8836e-06	1.8027e-07	3.5626e-08	1.1273e-08	4.6172e-09
2^{-5}	5.0014e-05	3.1282e-06	6.1801e-07	1.9561e-07	8.0141e-08
Method in [5]					
2^{-1}	5.0597e-04	2.5321e-04	1.6886e-04	1.2667e-04	1.0134e-04

2^{-2}	9.6556e-04	4.8357e-04	3.2250e-04	2.4194e-04	1.9357e-04
2^{-3}	1.7698e-03	8.8657e-04	5.9149e-04	4.4377e-04	3.5508e-04
2^{-4}	3.0307e-03	1.5201e-03	1.0145e-03	7.6132e-04	6.0926e-04
2^{-5}	4.7379e-03	2.3810e-03	1.5901e-03	1.1937e-03	9.5544e-04

Example 3: Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behavior,

$$\varepsilon^2 y''(x) + 0.25y(x - \delta) + y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \quad -\delta \leq x \leq 0, \quad y(1) = 0$$

The maximum absolute errors are presented in Table 5 for different values of δ . The graph of the computed solution for $\varepsilon = 0.003$ and different values of δ is also given in the Fig. 5.

Table 5. The maximum absolute errors of Example 3, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	N = 100	N = 200	N = 300	N = 400	N = 500
Present Method					
0.03	2.5455e-06	1.5921e-07	3.1447e-08	9.9499e-09	4.0754e-09
0.05	2.5297e-06	1.5805e-07	3.1219e-08	9.8790e-09	4.0466e-09
0.09	2.5301e-06	1.5813e-07	3.1244e-08	9.8831e-09	4.0487e-09
Method in[12]					
0.03	1.0936e-05	6.8420e-07	1.3515e-07	4.2761e-08	1.7516e-08
0.05	1.0707e-05	6.6910e-07	1.3217e-07	4.1826e-08	1.7132e-08
0.09	1.0117e-05	6.3260e-07	1.2499e-07	3.9546e-08	1.6197e-08
Method in[5]					
0.03	2.4582e-03	1.2196e-03	8.1096e-04	6.0742e-04	4.8554e-04
0.05	2.5127e-03	1.2472e-03	8.2948e-04	6.2134e-04	4.9669e-04
0.09	2.6198e-03	1.3016e-03	8.6589e-04	6.4872e-04	5.1863e-04

Example 4: Consider the singularly perturbed delay reaction-diffusion equation with oscillatory behavior,

$$\varepsilon^2 y''(x) + y(x - \delta) + 2y(x) = 1$$

under the interval and boundary conditions

$$y(x) = 1, \quad -\delta \leq x \leq 0, \quad y(1) = 0$$

The maximum absolute errors are presented in Table 6 for different values of δ . The graph of the computed solution for $\varepsilon = 0.003$ and different values of δ is also given in the Fig 6.

Table 6. The maximum absolute errors of Example 4, for different values of δ with $\varepsilon = 0.1$.

$\delta \downarrow$	N = 100	N = 200	N = 300	N = 400	N = 500
Present Method					
0.03	2.3125e-05	1.4440e-06	2.8519e-07	9.0243e-08	3.6967e-08
0.05	4.1761e-05	2.6084e-06	5.1553e-07	1.6312e-07	6.6808e-08
0.09	8.6141e-04	5.3757e-05	1.0616e-05	3.3585e-06	1.3756e-06
Method in[12]					
0.03	9.5107e-05	5.9430e-06	1.1739e-06	3.7154e-07	1.5220e-07
0.05	1.6132e-04	1.0082e-05	1.9930e-06	6.3066e-07	2.5831e-07
0.09	3.7561e-03	2.3545e-04	4.6501e-05	1.4716e-05	6.0275e-06
Method in[5]					
0.03	1.8682e-02	9.0640e-03	5.9795e-03	4.4608e-03	3.5572e-03
0.05	1.4987e-02	7.2328e-03	4.7631e-03	3.5505e-03	2.8299e-03
0.09	2.1346e-02	1.0306e-02	6.7863e-03	5.0577e-03	4.0307e-03

The Rate of Convergence (ρ): In the same way in equation (14) one can define $z_{h/2}$ by replacing h by $h/2$ and $N-1$ by $2N-1$, that is

$$z_{h/2} = \max_i |y_i^{h/2} - y_i^{h/4}|, \text{ for } i = 1, 2, \dots, 2N-1.$$

The computational Rate of convergence r is also obtained by using double mesh principle defined as [9].

$$r = \frac{(\log(z_h) - \log(z_{h/2}))}{\log 2}$$

The following table (Table 7) shows the rate of convergence r of the present method for different values of the mesh size h .

Table 7. Rate of convergence r for $\varepsilon = 0.1$ and $\delta = 0.05$.

h	$h/2$	Z_h	$h/4$	$Z_{h/2}$	r
Example 1					
1/100	1/200	2.1884e-07	1/400	1.3703e-08	3.9973
1/200	1/400	1.3703e-08	1/800	8.5688e-10	3.9993
1/300	1/600	2.7080e-09	1/1200	1.6930e-10	3.9996
Example 2					
1/100	1/200	9.8920e-08	1/400	6.1882e-09	3.9987
1/200	1/400	6.1882e-09	1/800	3.8679e-10	3.9999
1/300	1/600	1.2224e-09	1/1200	7.6362e-11	4.0007
Example 3					
1/100	1/200	2.5297e-06	1/400	1.5805e-07	4.0005
1/200	1/400	1.5805e-07	1/800	9.8790e-09	3.9999
1/300	1/600	3.1219e-08	1/1200	1.9515e-09	3.9998
Example 4					
1/100	1/200	4.1761e-05	1/400	2.6084e-06	4.0009
1/200	1/400	2.6084e-06	1/800	1.6312e-07	3.9992
1/300	1/600	5.1553e-07	1/1200	3.2217e-08	4.0002

The Effect of Delay Term on the Solution Profile

To analyze the effect of the delay term on the solution profile of the problem, the numerical solution of the problem for different values of the delay parameters have been given by the following graphs.

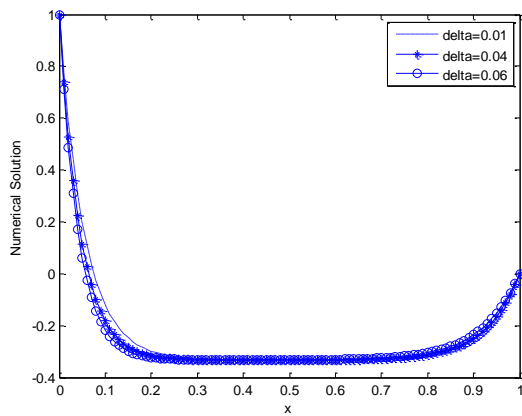


Fig. 1. The numerical solution of Example 1 with $\epsilon = 0.01$ and $N=100$.

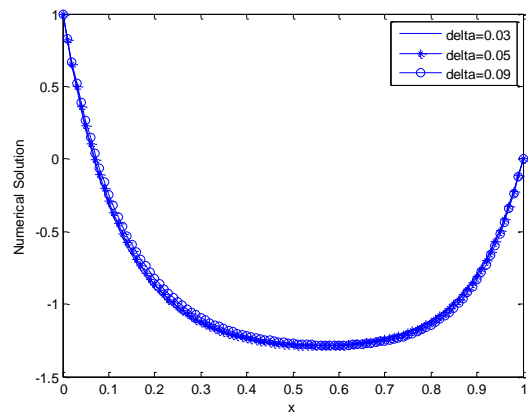


Fig. 2. The numerical solution of Example 2 with $\epsilon = 0.01$ and $N=100$.

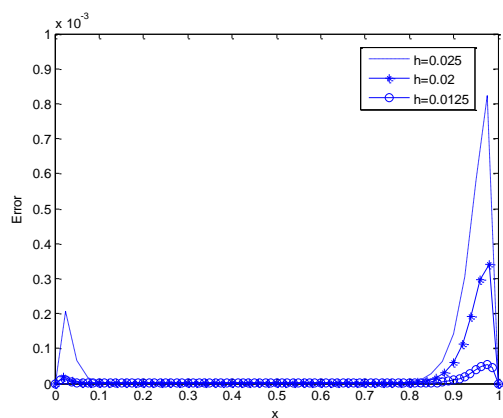


Fig. 3. The point-wise absolute errors of Example 1 for different values of mesh size h , $\epsilon=2^{-5}$ and $\delta = 0.5\epsilon$.

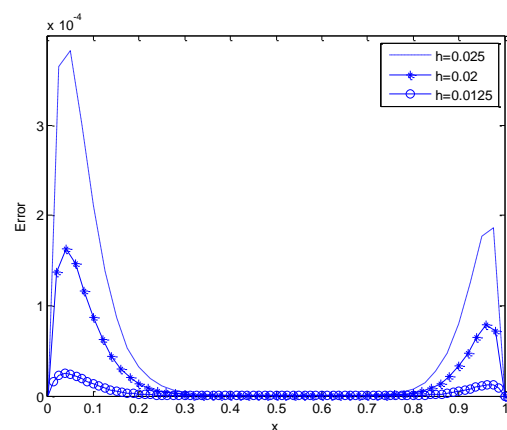
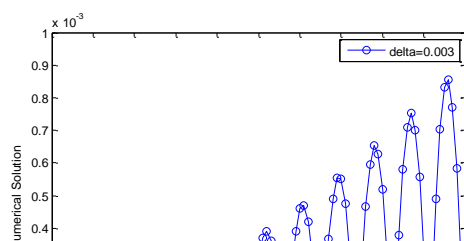
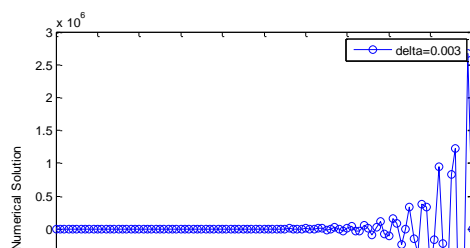


Fig. 4. The point-wise absolute errors of Example 2 for different values of mesh size h , $\epsilon=2^{-5}$ and $\delta = 0.5\epsilon$.



pl. N.



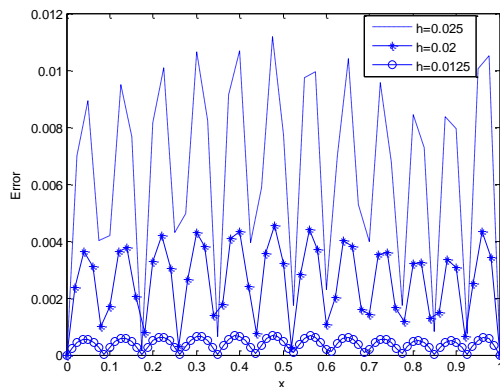


Fig. 7. The point-wise absolute errors of Example 3 for different values of mesh size h , $\varepsilon=2^{-5}$ and $\delta = 0.5\varepsilon$.

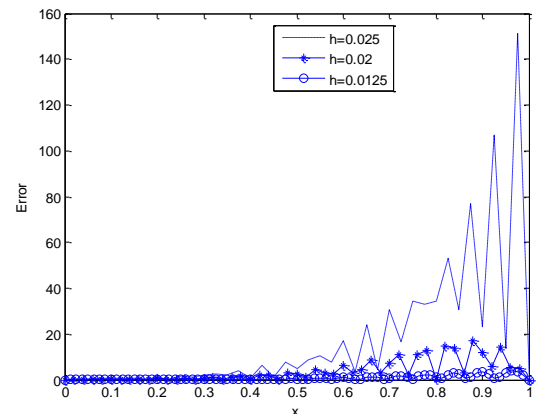


Fig. 8. The point-wise absolute errors of Example 4 for different values of mesh size h , $\varepsilon=2^{-5}$ and $\delta = 0.5\varepsilon$.

5. Results and Discussions

Fourth order fitted operator numerical method for solving singularly perturbed reaction-diffusion with delay has been presented. To demonstrate the efficiency of the method, four model examples without exact solutions have been considered for different values of the perturbation parameter ε and delay parameter δ and also results are presented in the Tables and Figures. It is observed that from the tables the present method approximate the solution and the stability and convergence of the method is established well. The effect of the delay on the solution of singularly perturbed delay reaction-diffusion equation is showed by plotting graphs of four model examples. Two model examples of with twin layers behavior and two model examples of with oscillatory layers have been considered and solved for different values of perturbation parameter ε , delay parameter δ and mesh size h . The numerical solutions are tabulated (Tables 1 to 6) in terms of maximum absolute errors and observed that the present method improves the findings of [5] and [12]. Also, it is significant that all of the maximum absolute errors decrease rapidly as N increases. The results presented in Table 7 confirmed that computational rate of convergence as well as the theoretical estimate indicates that the method is a fourth order convergent.

Further, to investigate the effect of delay on the solution of the problem, numerical solutions have been presented using graphs. Accordingly, when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behavior (Fig. 1). When the delay parameter is of $O(\varepsilon)$, the solution maintains layer behavior although the coefficient of the delay term in the equation of $O(1)$ and the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases (Fig. 2). For the oscillatory behavior case, one can conclude that the solution oscillates throughout the domain for different values of delay parameter δ (Fig. 4 and 5). In a concise manner, the present method gives more accurate solution and is uniformly convergent for solving singularly perturbed delay reaction-diffusion equations with twin layer and oscillatory behavior. Also it can see that as mesh size h decrease the absolute errors also decrease from (Fig. 3, 4, 7, 8).

Acknowledgements

The authors wish to express their thanks to Jimma University, College of Natural Sciences for technical support and the authors of literatures for the provided scientific aspects and idea for this work.

Conflict of interests

The authors declare that they have no conflict of interests.

References

- 1 Swamy D.K.; Phaneendra K.; Reddy Y.N. Accurate Numerical Method for Singularly Perturbed Differential-Difference Equations with Mixed Shifts. *Khayyam J. Math.*, 2018, **4**, 110-122. [[CrossRef](#)]
- 2 Longtin A.; Milton J.G. Complex Oscillations in the Human Pupil Light Reflex with Mixed and Delayed Feedback. *Math. Biosci.*, 1988, **90**, 183-199. [[CrossRef](#)]
- 3 Derstine M.W.; Gibbs H.M.; Hopf F.A.; Kaplan D.L. Bifurcation Gap in a Hybrid Optically Bistable System. *Phys. Rev. A.*, 1982, **26**, 3720-3722. [[CrossRef](#)]
- 4 Glizer V. Asymptotic Analysis and Solution of a Finite-Horizon h^∞ Control Problem for Singularly-Perturbed Linear Systems with Small State Delay. *J. Optim. Theory Appl.*, 2003, **117**, 295–325. [[CrossRef](#)]
- 5 Swamy D.K.; Phaneendra K.; Babu A.B.; Reddy Y.N. Computational Method for Singularly Perturbed Delay Differential Equations with Layer or Oscillatory Behavior. *Int. J. Appl. Math. Inform. Sci.*, 2016, **2**, 527-536. [[CrossRef](#)]
- 6 Phaneendra K.; Soujanya G.B.S.L.; Reddy Y.N. Numerical Solution of Second Order Singularly Perturbed Differential–Difference Equation with Negative Shift. *Int. J. Nonlin. Sci. Num.*, 2014, **18**, 200-209. [[CrossRef](#)]
- 7 Chakravarthy P.P.; Kumar S.D.; Rao R.N. An Exponentially Fitted Finite Difference Scheme for a Class of Singularly Perturbed Delay Differential Equations with Large Delays. *Ain Shams Eng. J.*, 2017, **8**, 663-671. [[CrossRef](#)]
- 8 Sirisha L.; Reddy Y.N. The Numerical Solution of Singularly Perturbed Differential Difference Equations with Dual Layer. *Am. J. Appl. Math. Stat.*, 2014, **2**, 336-343. [[CrossRef](#)]
- 9 Doolan E.R.; Miller J.J.H.; Schilders W.H.A. Uniform Numerical Methods for Problems with Initial and Boundary Layers. Boole Press, Dublin, 1980. [[Link](#)]
- 10 Kadalbajoo M.K.; Sharma K.K. Numerical Analysis of Singularly Perturbed Delay Equations with Layer Behavior. *Appl. Math. Comput.*, 2004, **157**, 11–28. [[CrossRef](#)]
- 11 Roos H.G.; Stynes M.; Tobiska L. Numerical Methods for Singularly Perturbed differential Equations. Springer, Berlin, 1996. [[Link](#)]
- 12 Gemechis F.; Gashu G.; Tesfaye A.; Reddy Y.N. Numerical Solution of Singularly Perturbed Delay Reaction-Diffusion Equations with Layer or Oscillatory Behavior. *Am. J. Num. Anal.*, 2017, **5**, 1-10. [[CrossRef](#)]
- 13 O'Malley R.E. Singular Perturbation Methods for Ordinary Differential Equations. Springer-Verlag, New York, 1991. [[Link](#)]
- 14 O'Riordan E.; Stynes M. A Globally Uniformly Convergent Finite Element Method for a Singularly Perturbed Elliptic Problem in Two Dimensions. *Math. Comput.*, 1991, **57**, 47-62. [[CrossRef](#)]
- 15 Clavero C.; Gracia J.L.; Jorge J.C. High-Order Numerical Methods for One Dimensional Parabolic Singularly Perturbed Problem with Regular Layers. *Numer. Meth. Part D E*, 2005, **21**, 149-169. [[CrossRef](#)]
- 16 Woldaregay M.M.; Duressa G.F. Parameter Uniform Numerical Method for Singularly Perturbed Differential Difference Equations. *J. Nigerian Math. Soc.*, 2009, **38**, 223-245. [[CrossRef](#)]
- 17 Swamy D.K.; Phaneendra K.; Babu A.B.; Reddy Y.N. Computational Method for Singularly Perturbed Delay Differential Equations with Twin Layers or Oscillatory Behavior. *Ain Shams Eng. J.*, 2015, **6**, 391-398. [[CrossRef](#)]



© 2020, by the authors. Licensee Ariviyal Publishing, India. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).