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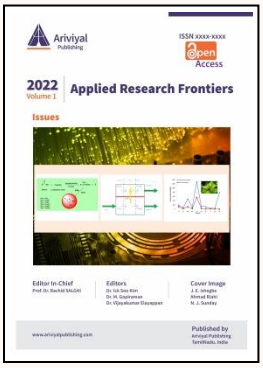

Existence of Weak Solution with Some Stochastic Hyperbolic Partial Differential Equation

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Abstract: The derivatives are well understood in suitable weak sense to make the space complete. Therefore, this paper studied stochastic hyperbolic Partial Differential Equation (PDE) in Sobolev spaces which proffered existence, Uniqueness of weak solution. The smoothness of weak solutions was analyzed to justify our computations. The estimates were logically combined and result showed to follow uniform hyperbolicity condition. To this end, energy estimates showed that the sequences were all bounded.

Keywords: Hyperbolic PDE; weak solutions; stochastic PDE and sobolev spaces

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1. Introduction

In mathematical finance generally, the application of Partial Differential Equation (PDE) cannot be over-emphasized due to its numerous uses in the society. For instance, Partial Differential Equations (PDEs) are used in modeling to analyze some dynamic systems in the areas such as physics, biology, and economics and finance etc. the Linear PDEs make up a large class of PDEs which is of a succinctly simple structure such that a thorough analysis is possible.[1] Hence, the dynamics of changing process can be modeled into ordinary differential equation or partial differential equation, depending on the nature of the problems, these equations may take various forms like ordinary differential equations, where partial differential equation (PDE) is an equation involving a function of several variables and at least, one of its partial derivatives and sometimes a combination of interacting equations of ordinary and partial differential equation. If some randomness is allowed into stochastic differential equation (SDE) for example; environmental effects are allowed into some of the coefficients of a differential equation, a more realistic mathematical model of the problem or situation can be obtained.[2]

However, mathematicians are given all serious attention to solution which has topological and analytical basis; such solutions exist and are not difficult to obtain. Hence, a sobolev space belongs to these families with redefinition of differentiability; and gives weak formulations to obtain weak solutions.

Nevertheless, good numbers of scholars have used PDEs in different approaches such as;

Mohammed E.R. et al.,[3] considered systems of nonlinear PDEs using Adomian and modified decomposition method. In another dimension Nwobi F.N.[4] studied the symmetry analysis of partial differential equation of hyperbolic type and result show the lie algebra is of $\{sl(2,R)\oplus w_3\}\oplus_3\infty A_1$ with 12 optimal systems. Company R. et al.,[5] employed the semi-discretization technique to deal with the issues arising as a result of a nonlinear case of interest modeling option pricing with transaction costs. Rao S.C.S.[6] also applied numerical scheme to the generalized Black-Scholes partial differential equation for European call option. The outcome showed that the second order accuracy in time and third order accuracy in space were obtained.

Denny D.[7] studied the existences of unique solution of an elliptic PDE. Result showed that the key proofs lies in obtaining a priori estimates. Panda A. et al.,[8] considered the existence of solution for a PDE involving a singularity with a general non negative radon measure as its nonhomogeneous terms. Canino A. et al.,[9] have considered quasilinear elliptic equation involving the p-laplacian and singular nonlinearities; they deduced a few comparison principles and have proved some unique results. In another dimension, DiPerna R.J. et al.,[10] obtained some new existence, uniqueness and stability results for ODEs with coefficients Sobolev spaces. In their results linear transport equations were analyzed by the method of renormalization solutions.

Osu B.O et al.,[11] Looked at solution to nonlinear Black Scholes equations. They proved the existence of weak solutions in a bounded domain and extend the results to the whole domain using a diagonal process. Osu B.O et al.,[12] Considered a weak solution of nonlinear Black Scholes Equation with transaction cost and portfolio risk in Sobolev space. In their result they obtained a weak solution that is characterized by Fourier transform. A few noteworthy papers on different PDEs can be found in references therein.[12-18]

This study is aimed at solving financial partial differential equation of hyperbolic type in Sobolev spaces on the basis of obtaining existence and uniqueness of weak solutions which have not been seen in this dynamic area of mathematical finance. This paper extends the work of Osu B.O et al.,[12] by considering stochastic hyperbolic PDE in such spaces.

The paper is arranged in the following ways: Section 2 mathematical frame work, Section 2.2 presents problem formulation, the definition, main results of Sobolev spaces are seen in 2.3. This paper is concluded in section 3.

2. Mathematical Frame work

2.1. Stochastic Processes

Definition 1 Stochastic process: A stochastic process $X(t)$ is a relation of random variables $\{X_t(\gamma), t\epsilon T, \gamma\epsilon\Omega\}$, i.e, for each t in the index set $T, X(t)$ is a random variable. Now we understand as time t and call $X(t)$ the state of the procedure at time t. In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors.

It can also be seen as a statistical event that evolves time in accordance to probabilistic laws. Mathematically, a stochastic process may be defined as a collection of random variables which are ordered in time and defines at a set of time points which may be continuous or discrete.

Definition 2: A stochastic process whose finite dimensional probability distributions are all Gaussian.(Normal distribution). Let

$$u(s,v,t) = e^{-rt}E[f\,(s,v,t)/S_0 = S, V_0 = V],\ f \in C^2c\,(\mathbb{R}^2,\mathbb{R}) \quad (1.1)$$

Where C^2c represents the function space of twice continuously differentiable functions with compact support, it $C^2c\,(\mathbb{R}^2,\mathbb{R}) = \{f \in C^2c(\mathbb{R}^2,\mathbb{R}) : lim_{/x/+/y/}f(x,y) = 0.\}$

Assuming that $P\,\epsilon\,[-1,1]$ is the correlation coefficient $\to\infty$ between the processes w^1t and w^2t, then the value $u(s,v,t)$ of the asset discounted at the rate r satisfies the partial differential equations as follows.[18]

$$\frac{\partial u}{\partial t} + VS^2\frac{\partial^2 u}{\partial S^2} + \rho\theta VS\frac{\partial^2 u}{\partial s\partial v} + \frac{1}{2}V\sigma^2\frac{\partial^2 u}{\partial V^2} + \gamma s\frac{\partial u}{\partial s} + \{k[\theta - V_t] - \lambda(s,v,t)\}\frac{\partial u}{\partial v} - \gamma u = o, t > 0 \quad (1.2)$$

$$u(s,v,t) = f(s,v) \quad (1.3)$$

In mathematical finance, for a contingent claim on a single asset, the generic PDE in the form.[14]

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u = 0 \quad (1.4)$$

Where t represents time to maturity, x represents the value of the underlying asset or some monotonic functions of it (e.g. log (S); log-spot) ad u is the value of the claim (as a function of x and t). The terms a(.), b(.) adc(.) and diffusion, convection and reaction Coefficients respectively .(1.4) can also be written as;

$$\frac{\partial u}{\partial t} + a(x,t)c\left(a(x,t)\frac{\partial}{\partial x}\right) + b(x,t)\frac{\partial}{\partial x}(\beta(x,t)u) + c(x,t)u = 0 \quad (1.5)$$

This PDE describes the evolution of the transition density of a stochastically quantity (e.g. a stock value) which occur in the Fokkper-Planoke (Kolmogrorov forward) equation. However our interest in this paper is the hyperbolic financial PDE that satisfies the following:

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{2}v\sigma\frac{\partial^2 u}{\partial v^2} + K[\theta - V_t]\frac{\partial u}{\partial v} - ru = 0 \quad (1.6)$$

2.2. Problem Formulation

Given: $U_T\ = U \times (O,T)\ where\ T > 0$ and $UC\mathbb{R}^N$ is an open bounded set. Following the initial /boundary-value problem

$$\begin{cases} U_u + L_u = f\ in\ U_T \\ u = o\ on\ \partial u \times [0,T] \\ U = g, u_t = h\ on\ U\ \times\ \{t = 0\} \end{cases} \quad (1.7)$$

Where $f\colon U_T \longrightarrow \mathbb{R},\ g,\ h{:}U \to \mathbb{R}\ are\ given,\ and\ U{:}U_T \to \mathbb{R}\ is$ the unknown $u = (x,t)$.

The symbol L denotes for each time t a second – order partial differential operator, having either the divergence form.

$$Lu = \sum_{ij=1}^{N}((v,t)^{ij}uv_i)\,v_i + \sum_{ij=1}^{N}\left(\frac{1}{2}(v\sigma^2)^{ij}(v,t)ur_i\right)\,v_j \quad (1.8)$$

$$+\sum_{i=1}^{N}K(\theta - V_t)^i\,(v_i)uv_i - (r(v_it)uv_i)u = 0$$

or else the non-divergence form

$$Lu = -\sum_{ij=1}^{N}(v,t)^{ij}\,uv_iu_j + \sum_{ij=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij}\,(v,t)uv_iu_j \quad (1.9)$$

$$+\sum_{i=1}^{N}K(\theta - V_t)^i\,(v,t)uu_i - r(v,t)u$$

For given coefficient $\left(\frac{1}{2}v\sigma^2\right)^{ij}, K(\theta - V_t)^i, r(i,j = 1,\dots,N)$

Definition: The partial differential operator $\frac{\partial^2}{\partial t^2} + L$ is said to be hyperbolic, if there exists α constant $\theta > o$ such that

$$\sum_{ij=1}^{N}\left(\frac{1}{2}v\sigma^2\right)^{ij}\,\xi_j\xi_j \geq \theta|\varepsilon|^2 \quad (1.10)$$

for all $(v,t) \in U_T\,, \xi j \in \mathbb{R}^N$

$\left(\frac{1}{2}v\sigma^2\right)^{ij} = \delta_{ij}$, $K(\theta - V_t)^i \equiv r \equiv f \equiv o$, then $L = -\Delta$ and u_{tt} wave equation.

2.3. Main Results of Sobolev spaces

Assuming L has U_{tt} divergence form (1.8) and look for an appropriate notion of weak solution for problem (1.7).
Suppose, initially that we have

$$\left(\frac{1}{2}v\sigma^2\right)^{ij}, K(\theta - V_t)^i, c\epsilon C^I(U_T) \ (i,j = 1,\dots,N) \quad (1.11)$$

$$f \in L^2(U_T), \quad (1.12)$$

$$g \in H_o^I(U), h \in L^2(U) \quad (1.13)$$

And always assume $\left(\frac{1}{2}v\sigma^2L^2\right)^{ij} = \left(\frac{1}{2}v\sigma^2\right)^{ij} (i,j = 1,\dots,N)$

We introduce bilinear form of time-dependent

$$B(u,v,t) := \int_u \sum_{ij=1}^{N} (v,t)^{ij}\, uvi\, vj + \int_u \sum_{ij=1}^{N} \left(\frac{1}{2}v\sigma^2\right)^{ij} (.,t) u_{(v,t)i}\, v_{(v,t)j} + \sum_{i=1}^{N} K(\theta - V_t)^i (.,t) u_{(v,t)i} v + v + r(.,t)uvd(v,t) \quad (1.14)$$

$for\ u,v \in H_0^1(U)$ and $O \leq t \leq T$.

Definition of weak solution: Suppose $u = u(v,t)$ to be a smooth solution of (1) and we defined the associated mapping as follows:
$u: [O,T] \rightarrow H_0^1(U)$
By
$[u(t)](v) := u(v,t)(v \in U, o \leq t \leq T)$
Similarly, we introduce the function
$f: [o,T] \rightarrow L^2(U)$
Defined by
$[f(t)](v) \coloneqq f(v,t), (v \in U, 0 \leq t \leq T)$, multiply the PDE $U_{tt} + L_u = f$
By v and integrate by parts, to obtain the identity below

$$(u^u u, v) + B[v,v;t] = (f,v) \quad (1.15)$$

For $0 \leq t \leq T$, *where* $(,)$ *represents the inner product in* $L^2(U)$.
We have seen from the PDE $U_{tt} + L_u = f$ such that

$$U_{tt} = g^o + \sum_{i=1}^{N} g^j_{(V,t)j}\ in\ U \quad (1.16)$$

$$for\ g^o = f - \sum_{i=1}^{N} k(\theta - v_t)^i \mathrm{U}_{(V,t)i} - ru\ and\ g^i : \sum_{i=1}^{N} \left(\frac{1}{2}v\sigma^2\right)^{ij} \mathrm{U}_{(V,t)i}$$

$(j = I, \dots, N)$. This implies looking for a weak solution u *with* $U^{II} \in H^{-I}(U)$ for $a.e.\ 0 \leq t \leq T$, *and so* reinterpret the first term of (1.15) as (U^{II}, v), (I) *representing* as usual the pairing between $H_0^1(U)$, is said to be weak solution of the hyperbolic initial/boundary – value problem
(i) for the fact that (i) $(u^u, v) + B[u,v,t] = (f,v)$
For each $v \in H_o^i(U)$ and a.e. time $o \leq t \leq T$, and
(ii) $u(o) = g, U^i(o) = h$

Existence of weak solutions (Galerkin approximation), we shall construct weak solution of the hyperbolic initial/boundary-value problem.

$$\begin{cases} U_{tt} + Lu = f\ in\ U_T \\ U = o\ on\ \partial u \times [O,T] \\ u = g, u_t = h\ on\ U \times \{t = o\} \end{cases}$$

In solving first, a finite dimensional approximate. We adopt the method of Galerkin's in selecting smooth functions $W_k = W_k(v)$ (17) $(k = 1,\dots)$ such that

$\{wk\}_{k=i}^{\propto}$ is an orthonormal basis of $H_o^I(U)$ (1.18)
$\{wk\}_{k=i}^{\propto}$ is an orthonormal basis of $L^2(U)$ (1.19)

Fixing a position integer M, we write

$$U_m(t) := \sum_{k=1}^{m} d_m^k(t) W_k \quad (1.20)$$

Where we select the coefficient $d_m^k(t)(0 \leq t \leq T, k = I,\dots,m)$

To satisfy $d_m^k(o) = (g, w_k)(k = I,\dots,m);$ (1.21)

$$d_m^k(o) = (h, w_k)(k = I,\dots,m); \quad (1.22)$$

And
$(U_m^1, w_k) + B[U_m, W_k; t] = (t, w_k)(o \leq t \leq T, k = I,\dots,m).$ (1.23)

Theorem 1 (construction of approximate solutions). For each interior $m = 1,2,\dots$, *this* exists a unique function U_m *of* the form (1.19) satisfying (1.20) (1.23).
Proof suppose U_m to the given by (1.19), we notice using (1.18).

$$U_m^u(t)wk = d_m^k(t) \quad (1.24)$$

We have

$$B[u_m, w_k; t] = \sum_{i=1}^{m} e^{kl}(t) d_m^1(t)$$

For $e^{kl}(t) := B[w_i, w_k; t](k, i, \dots, m)$, we also write $f^k(t) \coloneqq (f(t), w_k)(k = 1,\dots,m)$. Consequently (16) becomes the linear system of ODE

$$d_m^{k\,ll}(t) + \sum_{i=1}^{m} e^{kl}(t) d_m^l(t) = f^k(t)$$
$$(0 \leq t \leq T, k = l,\dots,m)^N$$

With the following initial condition (14), (15). According to `standard theory for ODEs, there exists a unique $C^2 function\ d_m(t) = (d_m^1(t), \dots, d_m^m(t)$, *satistying* (1.20), (1.21) and solving (18) for $0 \leq t \leq T$.

2.4. Energy Estimates

Here, we plan to send $m \to \infty$ and needs some estimates uniform in m.

Theorem 2 (Energy estimates). There exists a constant C, depending only on U, T and coefficients of L, such that

$$Max\ (||um(t)||H_o^i(u) + ||u_m^I(t)||L^2(U)t||U_m^{II}||L^2\{0,T;H^{-1}(U) \le C(\|f\|L^2(oT;\ L^2T(u)) + \|g\|H_o^I(U) + \|h\|L^2(U)) \quad (1.26)$$

For m = 1, 2

Proof

Multiply equality (16) by $d_m^k(t), sum\ k = I, \dots, m\ and$ recall (13) to discover the following

$$(u_m^{\iota\iota}, u_m^\iota) + B[u_m, u_m^{ij};\ t] = (f, u_m^\iota) \quad (1.27)$$

For a. e. $0 \le t \le T$. Notice that $(u_m^{11}, u_m^\iota)\frac{d}{dt}(\frac{1}{2}\|u_m^\iota\|^2L^2(u))$

Then write as follows:

$$B[u_m, u_m^i; t] = \int_u \sum_{ij=1}^N (v,t)^{ii}\, u(v,t);\ u_m^i(v,t);\ d(v,t) \quad (1.28)$$

$$+\int_u \sum_{ij=1}^N \left(\frac{1}{2}v\sigma^2\right)^{ij} u_m(v,t)_i u_m^1(v,t)_j d(v,t)$$

$$+\int_u \sum_{i=1}^N k(\theta - v_t)^i\, u_m(v,t)iu_m^i + ru_m u_m^i d(v,t)$$

$:= B_1 + B_2$

Since $\left(\frac{1}{2}v\sigma^2\right)^{ij} = \left(\frac{1}{2}v\sigma^2\right)^{ij}\ (i,j = 1,\dots,N)$

we see

$$B_1 = \frac{d}{dt}\left(\frac{1}{2}A[u_m.u_m,t]\right) - \frac{1}{2}\left|\int_u \sum_{i=1}^N \left(\frac{1}{2}v\sigma^2\right)^{ij} u_m(v,t)_i u_m(v,t)_j d(v,t)\right.$$

$$+\int_u \sum_{i=1}^N k(\theta - v_t)^i\, u_m(v,t)u_m^\iota + ru_m u_m^\iota d(v,t)$$

For the symmetric bilinear form

$$A[u,v,t] := \int_u \sum_{i=1}^N \left(\frac{1}{2}v\sigma^2\right)^{ij} u(v,t)_i v(v,t)_j d(v,t)\big(u, v \in H_o^\iota(U)\big) \quad (1.30)$$

The equality (1.29) implies

$$B_1 \ge \frac{d}{dt}\left(\frac{1}{2}A[u_m, u_m; t] - C(\|u_m\|^2 H_o^\iota(u)\right)$$

$$|B_2| \le C(\|u_m\|^2 H_o^l(u) + ||u_m^\iota||^2 L^2(u)$$

Combining the estimates (1.27) – (1.31) gives

$$\frac{d}{dt}\left(||u_m^l|\ \ |^2L^2(u)\right) + A[u_m, u_m, t] \le C(\|u_m^l|\ \ |^2L^2(u) + A[u_m, u_m; t] + ||f||^2L^2(u) \quad (25)$$

Where we used the inequality

$$\theta \int_u |Du|^2 d(v,t) \le A[u,v;t), (u \in H_o^i(U) \quad (1.33)$$

Which follows from the uniform hyperbolicity condition.

We write

$$\eta(t) := \|u_m^\iota(t)\|^2 L^2(u) + A\big[u_m(t), u_{m(t)j}\ t\big] \quad (1.34)$$

And

$$\xi(t):\ ||f(t)\|^2 L^2(u) \quad (1.35)$$

Then inequality (1.32)

$\eta'(t) \le c_1\eta(t) + C_2\xi(t)$

$for\ o \le t \le T$ and appropriate constant, C_1, C_2,

The grownwall's inequality yield the following estimate

$$\eta'(t) \le e^{c_1, t}\eta(0) + c_2\int_0^t(\xi(s)ds)\ (\ 0 \le t \le T) \quad (1.36)$$

Moreso, $\eta(0) = \|u_m^\iota(0)\|^2L^2(u) + A[U_m(0), u_m(0), 0] \le C\ (\|h\|^2L^2(u) + \|g\|^2H_o^\iota(u)\ )$

Following (1.20) and (1.21) we have the estimate

$\|u_m(0)\|H_o^l(u) \le \|g\|H_o^l(u)$. Thus formula (1.34) – (1.36)

Provided the bound

$\|u_m^1(t)\|^2L^2(u) + A[u_m(t), u_m(t);\ t]$
$\le C(\|g\|^2H_o^1(u) + \|h\|^2L^2(u) + \|f\|^2L^2(0,T;\ L^2(U))$

Since $0 \le t \le T$ was arbitrary, we see from this estimate and (1.33) that

$$\underset{0 \le t \le T}{\text{Max}}(\|um(t)\|^2H_o^1(u) + \|u_m^1(t)\|^2L^2(u))$$

$$\le (\|g\|^2H_o^l(u) + \|h\|^2L^2(u) + \|f\|^2L^2(0,T_j L^2(U))$$

$\Longrightarrow$ For any $v\epsilon H_m^1(u)$, $\|v\|H_o^1(u) \le 1$, and write $v = v^1 + v^2$

Where v^1t span $\{wk\}_{k=1}^m$ and $(v^2, wk) = 0\ (k = 1,\dots,m)$.

Note$\|v^\iota\|H_m^\iota(u) \le 1$. Then (1.19) and (1.23) imply

$$\langle u_m^u, v\rangle = (u_m^u, v) = (u_m^u,\ v^\iota) = (f, v^\iota) - B[u_m, v^\iota; t].$$

Thus

$|\langle u_m^u, v\rangle| \le C(\|f\|L^2(u) + \|u_m\|H_o^\iota(u))$

Since $\|v^\iota\|H_o^l(u) \le 1$. Consequently

$$\int_u^T \|\ u_m^u\|^2H^{-1}(u)dt \le C\int_o^T \|h\|^2L^2(u) + \|u_m\|^2H_o^l(u)dt$$

$$\le C(\|g\|^2H_o^l(u) + \|h\|^2L^2(u)(\theta,T;\ L^2(U))$$

2.5. Existence and uniqueness

We pass limits in our Galerkin appropriations

Theorem (Existence of weak solution). There exists a weak solution of (1.7).

Proof: From (1.26) of energy estimates, we see that the sequence $\{u_m\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;\ H_o^1(u), \{u_m^{\iota}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;\ L^2(u)$ and $\{u_m^{u}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;\ H^{-1}(u)$.
As a consequence there exists a subsequence $\{u_m\}_{i=1}^{\infty} \subset \{u_m\}_{m-1}^{\infty}$ *and* $u\epsilon L^2(0,T;\ H_o^1(u)$, with $u^{\iota}\epsilon$

$$L^2(O,T;L^2(u)), u^{\iota}\epsilon L^2, T; H^{-1}(U)$$

Such that

$$\begin{cases} u_{ml} \rightharpoonup u \text{ weakly in } L^2(O,T;L^2(u)) \\ u_{ml}^1 \rightharpoonup u^{\iota} \text{ weakly in } L^2(O,T;L^2(u)) \\ u_{ml}^{II} \rightharpoonup u^{\iota\iota} \text{ weakly in } L^2(O,T;H^{-1}(u)) \end{cases} \tag{1.37}$$

Next fix an interior N and choose a function $v\epsilon c^{\iota}([O,T];\ H_{\iota}^{\iota}(u)$ of the form

$$v\left(T = \sum_{K=1}^{N} d^k t w_k\right) \tag{1.38}$$

Where $\{d^k\}_{k=1}^N$ and smooth function. We select $m \geq N$, multiply (1.23) by $d^k(t)$, sum $k = 1, \ldots, N$ and the integrate with respect to t, to discover.

$$\int_o^1 \langle u_m^{\iota\iota}, v\rangle + B[u_m, v; t]dt = \int_o^T (f,v)dt$$

We set $m = mi$ and recall (1.37), to find in it unit that

$$\int_o^1 \langle u_m^{\iota\iota}, v\rangle + B[u, v; t]dt = \int_o^T (f,v)dt \tag{1.39}$$

We set $m = ml$ and recall (1.37), to find in the limit that

$$\int_o^1 \langle u_m^{\iota\iota}, v\rangle + B[u, v; t]dt = \int_o^T (t,v)dt \tag{1.40}$$

This equality then holds for all functions $v \in L^2(0,T;\ H_o^1(u)$, since functions of the form (1.38) are dense in this space. From (1.40) it follows further that

$$\langle u^{\iota\iota}, v\rangle + B[u,v;t] = (f,v)$$

For all $v\epsilon H_o^1(u)$ *and a.e.* $0 \leq t \leq T$. Furthermore, $u\epsilon C([0,T]; L^2(u))$ *and* $U\epsilon C([0,T]j\ H^{-1}(u))$
We verify

$$u(0) = g \tag{1.41}$$
$$u^{I}(0) = h \tag{1.42}$$

We choose any function say $v\epsilon C^2([0,T];\ H_o^I(u))$, with $v(T) = v^{\iota}(T) = 0$. Then integrating by parts twice w.r.t. t in (33) gives;

$$\int_o^T (v^{\iota\iota}, u) + B[u, v; t]dt$$
$$= \int_o^T (f,v)dt - \left(u(0), v^{\iota}(0)\right) + \langle u^{\iota}(0), v(0)\rangle$$

Similarly from (32) we deduce

$$\int_o^T (v^{\iota\iota}, u_m) + B[u_m, v; t]dt = \int_o^T (f,v)dt \tag{1.43}$$

$-(u_m(o), v^{\iota}(o) + (u_m^{\iota}(0), v(o))$
Setting $m = ml$ and recall (14), (15) and (30) to deduce

$$\int_o^T (v^{\iota\iota}, u) + B[u, v; t]dt = \int_o^T (f,v)dt - \left(g, v'(0) + \left(h, v(0)\right)\right) \tag{1.44}$$

Comparing identities (1.43) and (1.44). We can conclude that (1.41), (1.42), since $v(o), v^{\iota}(o)$ and arbitrary. Hence u is a weak solution of (1.7).

Theorem 4 (uniqueness of weak solution). A weak situation of 1 unique Proof.
It is enough to show that the only weak solution of (1) with $f \equiv g \equiv O$ *is*

$$u \equiv o \tag{1.45}$$

To verify this, fix $0 \leq s \leq T$ and set

$$v(t) := \begin{cases} \int_c^s u(t)dt \;\; if 0 \leq t \leq s \\ \qquad if\ s \leq t \leq T, \end{cases}$$

Then $v(t)\epsilon H_o^1(u)$ for each $0 \leq t \leq T$ *and so*

$$\int_o^s (u^{\iota\iota}, v)dt + B[u, v; t]dt = 0$$

Since $u'(0) = v(s) = 0$. We obtain after integrating by parts in the first form above.

$$\int_o^s (u^{\iota}, v^{\iota}) + B[u, v; t]dt = 0 \tag{1.46}$$

Now $v^{\iota} = -u(0 \leq t \leq s)$ and so

$$\int_o^s \langle u^{\iota\iota}, v\rangle + B[v^{\iota}, v; t]dt = 0$$

Thus:

$$\int_o^s \frac{d}{dt}\left(\frac{1}{2}\|u\|^2 L^2(u) - \frac{1}{2}B[v, v; t]dt\right) + \int_o^s C[u, v; t] + [v, v; t]\,dt$$

Where

$$C[u, v; t] := \int_u \sum_{i=1}^{N} k(\theta - v_i)^i v(v,t); u + \frac{1}{2}k(\theta - vt)_i (v,t)_i uvd(v,t)$$

And

$$D[u, v; t] := \frac{1}{2}\int_u \sum_{ij=1}^{N} (v,t)_{ij} u\,(v,t)_i v_{(v,t)_j} + \sum_{ij=1}^{N}\left(\frac{1}{2}v\sigma^2\right)_{ij}(v,t)_i\, v(v,t)$$
$$+ \sum_{ii-1}^{N} k(\theta - vt)_i, tu(s,v)iv + rtuvd(v,t)$$

For $u, v\ \epsilon H_o^1(u)$ Hence

$$\frac{1}{2}\|u(s)\|^2 L^2(u) + \frac{1}{2}B[v(o), v(0); t] = -\int_o^s C[u, v; t] + D[v, v; t]\,dt,$$

And consequently

$$\|u(s)\|^2 L^2(u) + \|v(o)\|^2 H_o^1(u)$$
$$\leq C\left(\int_o^s \|v\|^2 H_o^1(u) + \|u\|^2 L^2(u)dt + \|v(o)\|^2 L^2(u)\right) \tag{1.47}$$

We write:

$$w(t) := \int_0^t u(t)dt \;\; (o \leq t \leq T);$$

(1.47) now becomes

$$\|u(s)\|^2L^2(u) + \|w(s)\|^2H_o^1(u)$$
$$\leq C\left(\int_o^s \|w(t) - w(s)\|^2H_o^1(u) + \|u(t)\|^2L^2(u)dt + \|w(s)\|^2L^2(u)\right) \quad (1.48)$$

But $\|w(t) - w(s)\|^2H_o^1(u) \leq 2\|w(t)\|^2H_o^1(u) + 2\|w(s)\|^2H_o^1(u)$ and $\|w(s)\|L^2(u) \leq \int_o^s \|u(t)\| L^2(u)dt$

(1.48) implies

$$\|u(s)\|^2L^2(u) + (1 - 2_sC_1)\|w(s)\|^2H_o^1(u) \leq C_1 \int_o^s \|ws\|^2H_o^1(u) + \|u\|^2L^2(u)dt$$

We choose T_i so small that

$$1 - 2T_1C_1 \geq \frac{1}{2}$$

Then if $0 \leq s \leq T_1$ we have

$$\|u(s)\|^2L^2(u) + \|w(s)\|^2H_o^1 \leq C \int_o^s \|u\|^2L^2(u) + \|w\|^2H_o^1(u)dt$$

The integral form of grownwall's inequality, implies $u \equiv 0$ on $[0, T]$. We apply the same argument on the intervals $[T_1, 2T_1, 3T_1]$ etc eventually to deduce (1.45).

2.6. Regularity

In order to study the smoothness of our weak solution we will be motivated on the formal derivation of estimates.

(i) Suppose for the moment $u = (v, t)$ is a smooth solution of this initial – value problem for the wave equation:

$$U_{tt} - \Delta u = f \text{ in } \mathbb{R}^N \times (0, T)$$
$$u - g, u_i = h \text{ on } \mathbb{R}^N \times \{t = 0\}$$

And assume also u goes to zero as $/v/\to \infty$ sufficiently rapidly to justify the following calculation. We have

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N} /Du/^2 + U_t^2 d(v,t)\right) = 2\int_{\mathbb{R}^N} - Du_t + u_tu_{tt}d(v,t)$$
$$= 2\int_{\mathbb{R}^N} - u_t(u_t - \Delta u)d(v,t) = 2\int_{\mathbb{R}^N} - u_tfd(v,t)$$
$$\leq \int_{\mathbb{R}^N} U_t^2 d(v,t) + \int_{\mathbb{R}^N} f^2 d(v,t)$$

Applying Gronwall's inequality, we deduce

$$\sup_{u \leq t \leq T} \int_{\mathbb{R}^N} \|DU\|^e + U_t^e d(V_1t) \leq C \int_o^T \int_{\mathbb{R}^N} f^2 dvdt + \int_{\mathbb{R}^N} |Dg|^2 + h^2 d(v,t) \quad (1.49)$$

With the constant C depending only on T.

ii) We differentiate the PDE w.r.t and set $\bar{u} := ut$. Then

$$\begin{cases} \bar{u}_{tt} - \Delta\bar{u} = \bar{f} \text{ in } \mathbb{R}^N \times (0, T) \\ \bar{u} - \bar{g}, \bar{u}_t = \bar{h} \text{ on } \mathbb{R}^N \times \{T = 0\} \end{cases}$$

for $\bar{f} : f_t, \bar{g}: h, \bar{h} := u_{tt}(.,0) = f(.,0) + \Delta g$. *Applying estimate* (1.49) u, *we discover*

$$\sup_{u \leq t \leq T} \int_{\mathbb{R}^N} |Du_t|^2 + {u_{tt}}^2 d(v,t)$$
$$\leq C\left(\int_t^T \int_{\mathbb{R}^N} f_t^2 \, d(v,t) + \int_{\mathbb{R}^N} |D^2g|^2 + |Dh|^2\right) + f(.,0)^2 d(v,t) \quad (1.51)$$

$$\max_{\sigma \leq T \leq T} \| f(.,t)\|L^2(\mathbb{R}^N) \leq C\left(\|f\|L^2(\mathbb{R}^N \times (0,T)) + \|f_t\|L^2(\mathbb{R}^N \times (0,T))\right) \quad (1.52)$$

According to theorem 2,
Writing $-\Delta u = f - u_{tt}$
Deduce that

$$\left(\int_{\mathbb{R}^N} |D^2u|^2 d(v,t)\right) \leq C\int_{\mathbb{R}^N} f_t^2 + u_{tt}d(v,t)$$

For each $0 \leq t \leq T$
Combining (1.51) – (1.53) we can conclude that

$$\sup_{u \leq t \leq T} |D^2u|^2 + |Du_t|^2 + {u_{tt}}^2 d(v,t)$$
$$\leq C\left(\int_t^T \int_{\mathbb{R}^N} f_t^2 + f^2 dvdt + \int_{\mathbb{R}^N} |D^2g|^2 + |Dh|^2 dv\right)$$

The constant C depending only on T.

This estimates supports that bounds similar to (1.49) and (1.54) should be valid for our weak solution of a stochastic 2nd order hyperbolic PDE.

We calculate using the Galerkin approximation. To simplify the presentation, we however assume that $\{wk\}_{k=1}^{\infty}$ is the complete collection of open functions for $-\Delta$ on $H_o^1(u)$, and also that U is bounded open with ∂U smooth. In addition, Suppose

The coefficients $\left(\frac{1}{2}v\sigma^2\right)^{ij}, k(\theta - v_t)^i \, r(i, j = 1, \ldots, N)$ are smooth $\bar{u}$ and to not depend on t.

3. Conclusions

The analysis of stochastic hyperbolic PDE in Sobolev spaces has been perfectly demonstrated; showing the existence, uniqueness and smoothness of weak solution and other estimates that follow uniform hyperbolicity condition of the problem. Hence, the result of energy estimates revealed that sequences exists and is well bounded in the space. Finally we shall be looking at the financial implications of these PDEs in the next study.

Conflicts of Interest

The authors declare no conflict of interest.

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